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LEGENDRE-GAUSS-RADAU SPECTRAL COLLOCATION METHOD FOR NONLINEAR SECOND-ORDER INITIAL VALUE PROBLEMS WITH APPLICATIONS TO WAVE EQUATIONS*

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Abstract

We propose and analyze a single-interval Legendre-Gauss-Radau (LGR) spectral collocation method for nonlinear second-order initial value problems of ordinary differential equations. We design an efficient iterative algorithm and prove spectral convergence for the single-interval LGR collocation method. For more effective implementation, we propose a multi-interval LGR spectral collocation scheme, which provides us great flexibility with respect to the local time steps and local approximation degrees. Moreover, we combine the multi-interval LGR collocation method in time with the Legendre-Gauss-Lobatto collocation method in space to obtain a space-time spectral collocation approximation for nonlinear second-order evolution equations. Numerical results show that the proposed methods have high accuracy and excellent long-time stability. Numerical comparison between our methods and several commonly used methods are also provided.

Mathematics subject classification: 65M70, 41A10, 65L05, 35L05.
Key words: Legendre-Gauss-Radau collocation method, Second-order initial value problem, Spectral convergence, Wave equation.

1. Introduction

The initial value problems (IVPs) of second-order ordinary differential equations (ODEs) appear in many fields of science and engineering. In addition, a large number of second-order evolution equations, especially nonlinear wave equations, such as the Klein-Gordon and sine-Gordon equations, are often transformed into IVPs of second-order ODEs after appropriate spatial discretization methods. In the past few decades, great progress has been made in the study of numerical methods for the IVPs of (second-order) ODEs. Traditional and frequently used approaches for the numerical integration of (second-order) ODEs are mainly based on implicit and explicit finite difference, Runge-Kutta and Newmark-type schemes. We refer the reader to the monographs [8, 25, 26, 28, 29, 36, 41] for a comprehensive review.

As we all know, spectral methods have become important numerical methods for solving partial differential equations (PDEs), and have a wide range of applications in many fields of scientific and engineering computation, see, e.g., [6, 7, 10, 16, 18, 19, 40] and the references

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therein. Due to their high accuracy, spectral methods (including spectral collocation methods) have been applied to the numerical integration of ODEs in recent years. For example, Guo *et al.* developed several Laguerre spectral collocation methods [20, 23, 51, 52] and Legendre spectral collocation methods [21, 22, 24, 47] for nonlinear first and second-order IVPs of ODEs. In [1, 2], several spectral Galerkin and collocation methods were introduced for the numerical solutions of nonlinear Hamiltonian (ODE) systems. For some other high order methods (including the *hp*-version continuous and discontinuous Galerkin methods) for IVPs of ODEs, we refer the reader to [3, 39, 48, 49, 53] and the references therein.

The main purpose of the present paper is to introduce and analyze a new spectral collocation method based on Legendre-Gauss-Radau (LGR) points for the second-order ODEs of the form

$$\begin{cases} u''(t) = f(u'(t), u(t), t), & t \in (0, T], \\ u'(0) = v_0, & u(0) = u_0, \end{cases} \tag{1.1}$$

where the values v_0 and u_0 describe the initial states of $u(t)$ and f is a given function. For ease of statement, we sometimes use the notations $\partial_t u$ and $\partial_t^2 u$ instead of u' and u'', respectively.

We first design a single-interval spectral collocation scheme for problem (1.1) based on $N+1$ LGR points (see (2.21)). We then construct a simple but efficient iterative algorithm for numerical implementation of the single-interval collocation scheme by using Legendre polynomial expansion. We carry out a rigorous error analysis for the proposed scheme. Theoretical results show that the single-interval LGR collocation scheme has spectral accuracy, namely, for any fixed mode N, the smoother the exact solution is, the more accurate the numerical solution is. We also note that a Legendre-Gauss (LG) spectral collocation method has been proposed and analyzed in [24] for second-order ODEs. The main differences between the present paper and [24] are as follows: 1) Our collocation scheme is based on the LGR points, while the scheme in [24] is based on the LG points, this brings us different considerations in the theoretical analysis; 2) Due to different choices of the collocation points and different analysis approaches, the convergence of our method in N is half order higher than the method developed in [24] (see also Remark 2.2); 3) We design a new fixed-point iterative algorithm, which is much simpler and faster than that in [24] (see numerical comparison in subsection 2.5.2).

In order to improve the computational efficiency, we further propose a multi-interval LGR collocation method based on domain decomposition. Roughly speaking, we divide the solution interval $(0, T]$ into a series of non-overlapping subintervals, and then adopt the single-interval LGR collocation scheme and the corresponding iterative algorithm to obtain local approximation on each subinterval. The multi-interval LGR collocation scheme has the following advantages:

- For large T, we can obtain the numerical solution by the single-interval LGR collocation method on each subinterval step by step. In particular, the corresponding nonlinear algebraic system on each subinterval usually contains only a small number of unknowns. Therefore, the multi-interval collocation scheme can be implemented efficiently and economically. At the same time, it keeps the global spectral accuracy.

- The multi-interval LGR collocation scheme has great flexibility with respect to the local time steps and local approximation degrees. It is a variable-step and variable-order scheme. This feature makes it easy for us to deal with solutions with complex dynamic behaviors, such as oscillatory, singular and long-time behaviors.