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Standing Waves of Fractional Schrödinger Equations with Potentials and General Nonlinearities

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Abstract. We study the existence of standing waves of fractional Schrödinger equations with a potential term and a general nonlinear term:

$$\mathrm{i}u_t - (-\Delta)^s u - V(x)u + f(u) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where $s \in (0,1)$, $N > 2s$ is an integer and $V(x) \le 0$ is radial. More precisely, we investigate the minimizing problem with L^2-constraint:

$$E(\alpha) = \inf\left\{ \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 + V(x)|u|^2 - 2F(|u|) \,\middle|\, u \in H^s(\mathbb{R}^N),\ \|u\|^2_{L^2(\mathbb{R}^N)} = \alpha \right\}.$$

Under general assumptions on the nonlinearity term $f(u)$ and the potential term $V(x)$, we prove that there exists a constant $\alpha_0 \ge 0$ such that $E(\alpha)$ can be achieved for all $\alpha > \alpha_0$, and there is no global minimizer with respect to $E(\alpha)$ for all $0 < \alpha < \alpha_0$. Moreover, we propose some criteria determining $\alpha_0 = 0$ or $\alpha_0 > 0$.

Key Words: Fractional Schrödinger equation, standing wave, normalized solution.

AMS Subject Classifications: 35R11, 35A01, 35A15

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1 Introduction and main results

In this paper, we study standing waves of fractional Schrödinger equations with the potential term and general nonlinearity:

$$\mathrm{i}u_t - (-\Delta)^s u - V(x)u + f(u) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $N > 2s$ is an integer and the $(-\Delta)^s$ is the fractional Laplacian of the following form:

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy,$$

where $P.V.$ means the Cauchy Principle value on the integral and $C_{N,s}$ is some positive normalization constant, see [7,14] for details. If u is a standing wave, i.e., $u(t,x) = \mathrm{e}^{i\mu t} w(x)$, then $w \in H^s(\mathbb{R}^N)$ and $\mu \in \mathbb{R}$ satisfy the following equation:

$$(-\Delta)^s w + V(x)w - f(w) = -\mu w, \quad x \in \mathbb{R}^N. \tag{1.2}$$

We are interested in looking for solutions (w,μ) under the restriction $\|w\|^2_{L^2(\mathbb{R}^N)} = \alpha$.

The fractional nonlinear Schrödinger equation is first discovered by N. Laskin (see [26, 27]) as an extension of Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. For $\inf_{\mathbb{R}^N} V(x) > 0$, there has been a lot of results involving the existence of the ground states and bound states for $(-\Delta)^s u + V(x)u = f(x,u)$ in $\mathbb{R}^N$ by variational methods, see [2,4,19,21,34] and the references therein. We note that [2,4,19,21,34] do not consider normalized solutions. When $V(x) \equiv 1$, the uniqueness of radial solutions for $(-\Delta)^s u + u = u^{\alpha+1}$ is obtained by [17,18]. When $V(x)$ is periodic, the existence of ground state solutions is investigated in [1,23,32,39]. For the case $V(x)$ is allowed to be sign-changing, existence and multiplicity results of nontrivial solutions are given by [5,13,25].

As for normalized solutions, Cheng [9] requires $V(x)$ to be coercive besides $V(x) > 1$. Du-Tian-Wang-Zhang [16] assume that $0 \le V(x) \in L^\infty_{loc}(\mathbb{R}^N)$ is coercive and $\inf_{\mathbb{R}^N} V(x) = 0$. Guo-Huang [20] investigate the normalized solutions of (1.1) in the case $V(x) \equiv 0$ and $f = f(x,u)$. Even though the $f(x,u)$ seems more general than $f(u) - V(x)u$, they are quite different by the specific assumptions on f. Luo-Zhang [28] study the normalized solutions to

$$(-\Delta)^s u = \lambda u + \mu |u|^{q-2} u + |u|^{p-2} u.$$

Ikoma-Miyamoto [22] discuss the existence of standing waves of (1.1) with $s = 1$. In this paper, we discuss a totally different case $V(x) \le 0$ is radial.

We focus on the existence of standing waves for (1.1). Namely, solutions of (1.1 of the special form $u(t,x) = \mathrm{e}^{i\mu t} w(x)$, where $\mu \in \mathbb{R}$ and $w \in H^s(\mathbb{R}^N)$, where $H^s(\mathbb{R}^N)$ is the fractional Sobolev space (see [7,14]). Throughout the paper, denote

$$[u,v]_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))(\bar{v}(x)-\bar{v}(y))}{|x-y|^{N+2s}} dxdy, \quad [u]_{H^s(\mathbb{R}^N)} := [u,u]^{\frac{1}{2}}_{H^s(\mathbb{R}^N)}.$$