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LIE-POISSON NUMERICAL METHOD FOR A CLASS OF STOCHASTIC LIE-POISSON SYSTEMS

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Abstract. We propose a numerical method based on the Lie-Poisson reduction for a class of stochastic Lie-Poisson systems. Such system is transformed to SDE on the dual $\mathfrak{g}^*$ of the Lie algebra related to the Lie group manifold where the system is located, which is also the reduced form of a stochastic Hamiltonian system on the cotangent bundle of the Lie group by momentum mapping. Stochastic Poisson integrators are obtained by discretely reducing stochastic symplectic methods on the cotangent bundle to integrators on $\mathfrak{g}^*$. Stochastic generating functions creating stochastic symplectic methods are used to construct the schemes. An application to the stochastic rigid body system illustrates the theory and provides numerical validation of the method.

Key words. Stochastic Lie-Poisson systems, structure-preserving algorithms, Poisson integrators, Lie-Poisson reduction, Poisson structure, Casimir functions.

1. Introduction

Stochastic Poisson systems (SPSs) are stochastic differential equation systems (SDEs) of the following form ([12]):

$$\begin{aligned} dy(t) &= B(y(t))\left(\nabla H_0(y(t))dt + \sum_{r=1}^{s} \nabla H_r(y(t)) \circ dW_r(t)\right), \\ y(0) &= y_0, \end{aligned} \tag{1}$$

where $y_0 \in \mathbb{R}^m$, $H_r : \mathbb{R}^m \to \mathbb{R}$ $(r = 0, \ldots, s)$ are smooth functions, $\{\mathcal{W}_r(t)\}_{t\geq 0}$ $(r = 0, \ldots, s)$ are independent standard real valued Wiener processes defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, '$\circ$' indicates that the SDEs are of Stratonovich sense. $B(y) = (b_{ij}(y))$ is called the structure matrix of the SPS, which is is a smooth $m \times m$ matrix–valued function of y with the skew-symmetry $b_{ij}(y) = -b_{ji}(y)$, and satisfies

$$\sum_{l=1}^{m} \left(\frac{\partial b_{ij}(y)}{\partial y^l} b_{lk}(y) + \frac{\partial b_{jk}(y)}{\partial y^l} b_{li}(y) + \frac{\partial b_{ki}(y)}{\partial y^l} b_{lj}(y) \right) = 0, \tag{2}$$

for all $i, j, k \in \{1, \ldots, m\}$. These properties of $B(y)$ guarantee that it induces the Poisson bracket of two smooth functions $K(y)$ and $L(y)$ by

$$\{K, L\}(y) = \nabla K(y)^T B(y) \nabla L(y), \tag{3}$$

which satisfies the skew-symmetry, Jacobi identity and the Leibniz' rule, as the case for canonical Poisson bracket of Hamiltonian systems ([9]).

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In this sense, the SPSs can be considered as generalizations of stochastic Hamiltonian systems (SHSs) ([5, 12, 22]) :

$$\begin{aligned} dy(t) &= J^{-1}\left(\nabla H_0(y(t))dt + \sum_{r=1}^{s} \nabla H_r(y(t)) \circ dW_r(t)\right), \\ y(0) &= y_0, \end{aligned} \tag{4}$$

where $J^{-1} = \begin{pmatrix} \mathbf{0}_d & -\boldsymbol{I}_d \\ \boldsymbol{I}_d & \mathbf{0}_d \end{pmatrix}$ and $\boldsymbol{I}_d$ is the d-dimensional identity matrix. When the dimension of a SPS is even, i.e. $m = 2d$, and $B(y) \equiv J^{-1}$, the SPS degenerates to a SHS. If the diffusion part vanishes, i.e. $\nabla H_r \equiv \mathbf{0}$, (1) are deterministic Poisson systems which have got attention since 19th century (see e.g. [9] and references therein). The Poisson and Hamiltonian systems can transform to each other by coordinate transformations or Poisson reductions ([9, 18] and references therein). Numerical methods for SPSs can be constructed using these properties, such as those based on the Darboux-Lie theorem ([9]) which transform symplectic methods for SHSs to Poisson integrators for SPSs via coordinate transformations ([12]). In this paper, however, we attempt another way, to construct Poisson integrators via Poisson reductions for a class SPSs, generalizing the deterministic Lie-Poisson reduction numerical approach ([4, 9, 13, 26]) to stochastic cases.

Almost surely, the phase flow of the SPS (1) $\varphi_{t,\omega} : y \to \varphi_{t,\omega}(y)$ possesses the Poisson structure, i.e. ([3, 12])

$$\frac{\partial \varphi_{t,\omega}(y)}{\partial y} B(y) \frac{\partial \varphi_{t,\omega}(y)}{\partial y}^T = B(\varphi_{t,\omega}(y)), \quad \forall t \geq 0, \quad a.s. \tag{5}$$

If the rank of $B(y)$ is not full such that there exist functions $C(y)$ yielding

$$B(y)\nabla C(y) = \mathbf{0}$$

almost surely, then these functions are called Casimir functions ([9]) of the SPSs, which are invariants of the systems, since almost surely

$$dC(y) = \nabla C(y)^T dy = \nabla C(y)^T B(y) \left(\nabla H_0(y)dt + \sum_{r=1}^{s} \nabla H_r(y) \circ dW_r(t)\right) = 0.$$

Now we consider special structure matrices $B(y)$ whose elements depend linearly on y, i.e.

$$b_{ij}(y(t)) = \sum_{k=1}^{m} C_{ji}^k y^k(t), \quad \forall i, j = 1, \ldots, m. \tag{6}$$

Analog to deterministic case ([9]), SPSs (1) with $B(y)$ fulfilling (6) are called stochastic Lie-Poisson systems (SLPSs) ([3, 11, 16]). The skew-symmetry as well as properties (2) and (6) of $B(y)$ make it possible to define a Lie bracket calculation using the constants C_{ij}^k in (6) by:

$$[E_i, E_j] = \sum_{k=1}^{m} C_{ij}^k E_k, \quad i, j = 1, \ldots, m \tag{7}$$

on a vector space with basis $\{E_i\}$ $(i = 1, \ldots, m)$. The vector space equipped with the Lie bracket calculation constituts a Lie algebra ([9]), denoted by $\mathfrak{g}$.

Lie-Poisson systems arise in celestial mechanics, robotics, fluid mechanics, and rigid body, etc. Typical examples include the Vlasov-Poisson equations, the Euler equations for rigid bodies ([4, 9, 13, 18, 19]). Numerical methods for deterministic Lie-Poisson systems have been developed during the last decades, including the