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STABILITY AND CONVERGENCE OF STEPSIZE-DEPENDENT LINEAR MULTISTEP METHODS FOR NONLINEAR DISSIPATIVE EVOLUTION EQUATIONS IN BANACH SPACE*

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Abstract

Stability and global error bounds are studied for a class of stepsize-dependent linear multistep methods for nonlinear evolution equations governed by ω-dissipative vector fields in Banach space. To break through the order barrier $p \leq 1$ of unconditionally contractive linear multistep methods for dissipative systems, strongly dissipative systems are introduced. By employing the error growth function of the methods, new contractivity and convergence results of stepsize-dependent linear multistep methods on infinite integration intervals are provided for strictly dissipative systems ($\omega < 0$) and strongly dissipative systems. Some applications of the main results to several linear multistep methods, including the trapezoidal rule, are supplied. The theoretical results are also illustrated by a set of numerical experiments.

Mathematics subject classification: 65J15, 65M12, 65M15, 65J08.
Key words: Nonlinear evolution equation, Linear multistep methods, ω-dissipative operators, Stability, Convergence, Banach space.

1. Introduction

In this paper, we are concerned with the time discretization of the nonlinear evolution equations

$$u'(t) = \mathcal{A}\big(t, u(t)\big), \quad t \in I_T = [0, T]; \qquad u(0) = u_0, \tag{1.1}$$

where $T > 0$ is a constant, $u : I_T \to X$, $u_0 \in X$ and the nonlinear map $\mathcal{A}$ is ω-dissipative in Banach space X. Such type of equations is found in a wide range of applications and typical examples of nonlinear map $\mathcal{A}$ are the porous medium vector field $\Delta(|u|^{\alpha-1}u)$ and the α-Laplacian $\nabla\cdot(|\nabla u|^{\alpha-2}\nabla u)$ (for more examples, see Section 2 and [4,5,13,23,31,34,38]). Much work has been devoted to time discretizations of nonlinear evolution equations, especially, in a finite-dimensional space or in a real-valued infinite-dimensional Hilbert space; see, e.g., [1–3, 14,15,19–21,28,29,43]. In Banach space, the studies of time discretizations have predominantly considered the backward Euler method; see, e.g., [8–10, 17, 22, 26, 31, 37]. This arises partly because the convergence of this type of discretization approximations can be used to show the existence of the solutions to the nonlinear evolution equations (see, e.g., [5, 6, 8, 12]). Perhaps the most reason is a well-known fact that unconditionally contractive methods in Banach space are subject to an order barrier $p \leq 1$; see, e.g., [42]. To obtain high order contractive methods

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in Banach space, circle condition was introduced and conditional contractivity was considered; see, e.g., [25, 35, 42, 48].

On the one hand, only first order linear multistep methods and Runge-Kutta methods are unconditionally contractive for dissipative systems in Banach space. On the other hand, there exist some second order linear multistep methods which are unconditionally contractive for dissipative systems in Hilbert space. The question arises whether there is a reasonable problem class between the above two classes of problems such that some second order linear multistep methods are unconditionally contractive for this class of problem. A natural approach would be to impose a restrictive condition on dissipative systems in Banach space. To this end, we introduce a quasi-reversible condition such that the problems class considered is a subset of nonlinear ω-dissipative systems in Banach space. Of particular interest in this condition is that it naturally holds for nonlinear dissipative systems in Hilbert space. Alternatively, the class of the dissipative problems in Hilbert space is a subset of the problem class considered here.

We note that high order linear multistep methods and Runge-Kutta methods for nonlinear parabolic problems in Banach space have been dealt with by linearization in [32] and [33], respectively, under the assumption that the linearization of the vector field is a sectorial map. Using the theory of logarithmic Lipschitz constants, by approximating the nonlinear semigroup $e^{t\mathcal{A}}(u_0)$, the convergence of strongly A-stable linear multistep methods for strictly dissipative differential systems, i.e., $\omega < 0$, was obtained in Hilbert space [21]. Here we will explore a different approach and generalize the classical B-theory [14, 16, 19, 28] for numerical methods for ordinary differential equations (ODEs) to linear multistep approximations of evolution equations in Banach space. This is done by extending the error growth function introduced by Burrage and Butcher [7] for implicit Runge-Kutta methods; see, also, [18, 19]. By computing the error growth function, the stability and the convergence of stepsize-dependent linear multistep (SDLM) methods for ω-dissipative systems are established. The strict-contractivity and the long time convergence of a class of SDLM methods on semi-infinite intervals for strictly or strongly dissipative differential systems will be also obtained.

The paper is organized as follows. We start in Section 2 by introducing some definitions and notations relative to the ω-dissipative problems. In Section 3, SDLM methods are proposed to solve the nonlinear evolution equations (1.1). In this section the existence and uniqueness of the solution to the discrete systems is provided. Section 4 is devoted to stability analysis of SDLM methods for this class of equations. The global error bounds are derived in Section 5. Especially, the long-time error bounds are obtained for a class of SDLM methods for strictly or strongly dissipative systems in this section. Some applications of the main results are postponed to Section 6. A numerical study is carried out for a test case in Section 7. Section 8 contains a few concluding remarks.

2. ω-dissipative Systems in Banach Space

Let X be a real-valued Banach space equipped with the norm $\|\cdot\|$ and $\mathfrak{D}$ be some subset of X. We assume that the nonlinear operator $\mathcal{A} : I_T \times \mathfrak{D} \to X$ in (1.1) is an ω-dissipative operator [27, 44], whose definition is given by the following.

Definition 2.1 ([27, 44]). *An operator $\mathcal{A}$ is said to be dissipative if, for any $t \in I_T$,*

$$\|u - v\| \le \|u - v - \tau[\mathcal{A}(t, u) - \mathcal{A}(t, v)]\|, \quad \tau > 0, \quad u, v \in \mathfrak{D},$$