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A SCALAR AUXILIARY VARIABLE (SAV) FINITE ELEMENT NUMERICAL SCHEME FOR THE CAHN-HILLIARD-HELE-SHAW SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS*

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Abstract

In this paper, we consider the Cahn-Hilliard-Hele-Shaw (CHHS) system with the dynamic boundary conditions, in which both the bulk and surface energy parts play important roles. The scalar auxiliary variable approach is introduced for the physical system; the mass conservation and energy dissipation is proved for the CHHS system. Subsequently, a fully discrete SAV finite element scheme is proposed, with the mass conservation and energy dissipation laws established at a theoretical level. In addition, the convergence analysis and error estimate is provided for the proposed SAV numerical scheme.

Mathematics subject classification: 65N06, 65B99.
Key words: Cahn-Hilliard-Hele-Shaw system, Dynamic boundary conditions, Bulk energy and surface energy, Scalar auxiliary variable formulation, Energy stability, Convergence analysis.

1. Introduction

The Cahn-Hilliard-Hele-Shaw system (CHHS) has attracted more and more attentions in recent years, since this model describes two phase flows in a simple way. This system turns out to be the basic diffusion interface model for incompressible binary fluids confined in a Hele-Shaw cell [42, 43, 50], and it has been proposed to simplify the well-known Cahn-Hilliard-Navier-Stokes model, where the Navier-Stokes system is coupled with the convective Cahn-Hilliard equation [19, 38, 39, 59]. This model has also been used to describe spinodal decomposition of a binary fluid in a Hele-Shaw cell [33], tumor growth and cell sorting [25, 64], and two phase flows in porous media [17], etc.

The CHHS system with Neumann boundary conditions has been extensively studied in the existing literature [8, 9, 12, 31, 33, 49, 63]. On the other hand, the homogeneous Neumann boundary condition turns out to unsatisfactory in some cases, due to the fact that this simple boundary condition set-up ignores the effects of certain process on the boundary to the bulk dynamics; in other words, separate chemical reactions on the boundary are not taken into consideration. Nevertheless, in certain applications such as fluid dynamics and contact line problems, a more accurate description of the short-range interaction of the binary mixture with

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the solid wall of the vessel turns out to be necessary. At present, various dynamic boundary conditions have been derived and analyzed for the Cahn-Hilliard equation [5, 40, 41, 52], while the associated analysis for the CHHS system is very limited.

Let $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3$) be a bounded domain with a boundary $\Gamma := \partial\Omega$. The unit outer normal vector on Γ will be denoted by $\mathbf{n} = \mathbf{n}(x)$. The standard CHHS system is formulated as

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) - \epsilon \Delta \mu = 0 \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\mu + \epsilon \Delta \phi - f(\phi) = 0 \quad \text{in } \Omega \times (0, T], \tag{1.2}$$

$$\mathbf{u} + \nabla p + \gamma \phi \nabla \mu = 0 \quad \text{in } \Omega \times (0, T], \tag{1.3}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T], \tag{1.4}$$

where $\gamma > 0$ is a dimensionless surface tension parameter, $\mathbf{u}$ is the advective velocity, and p is the pressure. To describe a mixture of two materials, the phase field variable ϕ stands for the difference of two local relative concentrations. In more details, $\phi(x)$ $(x \in \Omega)$ takes the distinct values, 1 and -1, in the respective pure phases of the materials, while $\{x \in \Omega : -1 < \phi(x) < 1\}$ matches with the diffuse interface between them, whose thickness is proportional to the very small positive constant ϵ. The variable μ stands for the chemical potential in the bulk, which can be derived from the Fréchet derivative [16] of the following Ginzburg-Landau free energy

$$E_{bulk}[\phi] = \int_\Omega \left(\frac{\epsilon}{2} \left| \nabla \phi \right|^2 + F(\phi) \right) dx,$$

where the functional F denotes the bulk potential and $f(\phi) = F'(\phi)$. Typically, F has a double well form, which reaches its global minima at $\phi = \pm 1$ and a local maximum at $\phi = 0$.

The homogeneous Neumann boundary conditions corresponding to the system (1.1)-(1.4) are given by

$$\partial_n \phi = 0 \quad \text{on } \Gamma \times (0, T], \tag{1.5}$$

$$\partial_n \mu = 0 \quad \text{on } \Gamma \times (0, T], \tag{1.6}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T]. \tag{1.7}$$

However, especially for certain materials in the bounded region, boundary condition (1.5) is not well-pleasing, since certain additional effects of the boundary to the bulk dynamics are ignored. Meanwhile, several dynamic boundary conditions have been proposed in the existing literatures [18, 46, 47, 53, 65], to replace the homogeneous Neumann condition. In order to improve this phenomenon and to better describe the whole system, physicists put forward a surface free energy

$$E_{surf}[\phi_\Gamma] = \int_\Gamma \left(\frac{\kappa \epsilon}{2} \left| \nabla_\Gamma \phi_\Gamma \right|^2 + G(\phi_\Gamma) \right) dS,$$

where ∇_Γ denotes the surface gradient operator on Γ and G is a surface potential. Furthermore, $\kappa > 0$ is related to the effects of surface diffusion. Some numerical works [2, 3, 52] have been reported as well.

The total free energy corresponding to the dynamic boundary conditions becomes

$$E = E_{bulk}[\phi] + E_{surf}[\phi_\Gamma]. \tag{1.8}$$