Journal of Computational Mathematics
Vol.42, No.2, 2024, 570–596.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2204-m2021-0311

A REDUCED ORDER SCHWARZ METHOD FOR NONLINEAR MULTISCALE ELLIPTIC EQUATIONS BASED ON TWO-LAYER NEURAL NETWORKS*

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Abstract

Neural networks are powerful tools for approximating high dimensional data that have been used in many contexts, including solution of partial differential equations (PDEs). We describe a solver for multiscale fully nonlinear elliptic equations that makes use of domain decomposition, an accelerated Schwarz framework, and two-layer neural networks to approximate the boundary-to-boundary map for the subdomains, which is the key step in the Schwarz procedure. Conventionally, the boundary-to-boundary map requires solution of boundary-value elliptic problems on each subdomain. By leveraging the compressibility of multiscale problems, our approach trains the neural network offline to serve as a surrogate for the usual implementation of the boundary-to-boundary map. Our method is applied to a multiscale semilinear elliptic equation and a multiscale p-Laplace equation. In both cases we demonstrate significant improvement in efficiency as well as good accuracy and generalization performance.

Mathematics subject classification: 65N55, 35J66, 41A46, 68T07.
Key words: Nonlinear homogenization, Multiscale elliptic problem, Neural networks, Domain decomposition.

1. Introduction

Approximation theory plays a key role in scientific computing, including in the design of numerical PDE solvers. This theory prescribes a certain form of ansatz to approximate a solution to the PDE, allowing derivation of an algebra problem whose solution yields the coefficients in the ansatz. Various methods are used to fine-tune the process of translation to an algebraic problem, but the accuracy of the computed solution is essentially determined by the the underlying approximation theory. New approximation methods have the potential to produce new strategies for numerical solution of PDEs.

* Received November 3, 2021 / Revised version received February 16, 2022 / Accepted April 14, 2022 / Published online March 28, 2023 /
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During the past decade, driven by some remarkable successes in machine learning, neural networks (NNs) have become popular in many contexts. They are extremely powerful in such areas as computer vision, natural language processing, and games [45, 58]. What kinds of functions are well approximated by NNs, and what are the advantages of using NNs in the place of more traditional approximation methods? Some studies [11, 30, 57] have revealed that NNs can represent functions in high dimensional spaces very well. For Barron functions, in particular, unlike traditional approximation techniques that require a large number of parameters (exponential on the dimension), the number of parameter required for a NN to achieve a prescribed accuracy is rather limited. In this sense, NN approximation overcomes the "curse of dimensionality". This fact opens up many possibilities in scientific computing, where the discretization of high dimensional problems often plays a crucial role. One example is problems from uncertainty quantification, where many random variables are needed to represent a random field, with each random variable essentially adding an extra dimension to the PDE [9,44,79,80]. Techniques that exploit intrinsic low-dimensional structures can be deployed on the resulting high-dimensional problem [12, 15, 25, 41, 48]. Another example comes from PDE problems in which the medium contains structures at multiple scales or is highly oscillatory, so that traditional discretization techniques require a large number of grid points to achieve a prescribed error tolerance. Efficient algorithms must then find ways to handle or compress the many degrees of freedom.

Despite the high dimensionality in these examples, successful algorithms have been developed, albeit specific to certain classes of problems. With the rise of NN approximations, with their advantages in high-dimensional regimes, it is reasonable to investigate whether strategies based on NNs can be developed that may even outperform classical strategies. In this paper, we develop an approach that utilizes a two-layer NN to solve multiscale elliptic PDEs. We test our strategy on two nonlinear problems of this type.

The use of NN in numerical PDE solvers is no longer a new idea. Two approaches that have been developed are to use NN to approximate the solutions [14, 28, 32, 53, 59, 66, 75, 76, 82] or the solution map [10, 26, 38–40, 54, 55, 60–62, 67, 68, 77, 78, 81]. Due to the complicated and unconventional nature of approximation theory for NN, it is challenging to perform rigorous numerical analysis, though solid evidence has been presented of the computational efficacy of these approaches.

The remainder of our paper is organized as follows. In Section 2 we formulate the multiscale PDE problem to be studied. We give an overview of our domain decomposition strategy and the general specification of the Schwarz algorithm. In Section 3, we discuss our NN-based approach in detail and justify its use in this setting. We then present our reduced-order Schwarz method based on two-layer neural networks. Numerical evidence is reported in Section 4. Two comprehensive numerical experiments for the semilinear elliptic equation and the p-Laplace equation are discussed, and efficiency of the methods is evaluated. We make some concluding remarks in Section 5.

2. Domain Decomposition and the Schwarz Method for Multiscale Elliptic PDEs

We start by reviewing some key concepts. Section 2.1 describes nonlinear multiscale elliptic PDEs and discussed the homogenization limit for highly oscillatory medium. Section 2.2 outlines the domain decomposition framework and the Schwarz iteration strategy.