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MULTIRATE TIME ITERATIVE SCHEME WITH MULTIPHYSICS FINITE ELEMENT METHOD FOR A NONLINEAR POROELASTICITY*

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Abstract

In this paper, a multirate time iterative scheme with multiphysics finite element method is proposed and analyzed for the nonlinear poroelasticity model. The original problem is reformulated into a generalized nonlinear Stokes problem coupled with a diffusion problem of a pseudo pressure field by a new multiphysics approach. A multiphysics finite element method is adopted for the spatial discretization, and the generalized nonlinear Stokes problem is solved in a coarse time step and the diffusion problem is solved in a finer time step. The proposed algorithm is a decoupled algorithm, which is easily implemented in computation and reduces greatly computation cost. The stability analysis and the convergence analysis for the multirate iterative scheme with multiphysics finite element method are given. Some numerical tests are shown to demonstrate and validate the analysis results.

Mathematics subject classification: 65N30, 65N12.
Key words: Nonlinear poroelasticity model, Multiphysics finite element method, Multirate iterative scheme.

1. Introduction

Poromechanic is a fluid-solid interaction system at pore scale, which is a branch of continuum mechanics and acoustics that studies the behavior of fluid-saturated porous materials. If the solid is an elastic material, then the subject of the study is known as poroelasticity, one can go back to the works of Biot [2, 3], Terzhagi [29] and Coussy [16] for details. The field of poroelasticity is of increasing importance today in science and engineering fields, such as materials science, agricultural science, environmental engineering, petroleum engineering and bio-mechanical engineering and so on [8,12,22]. Moreover, the elastic material may be governed by linear or nonlinear constitutive law, which then leads respectively to linear and nonlinear poroelasticity. In the numerical computing, a big challenge is that there may exists numerical oscillation for pore pressure, which is referred as locking phenomenon [27]. For linear poroelasticity model, a vast number of numerical methods have been developed in recent years, such as finite element methods (FEMS), discontinuous Galerkin (DG) methods, hybrid high-order methods, and weak Galerkin methods, one can see [4, 13, 14, 24, 25] and the references therein

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for details. In this paper, we are focus on the numerical approximation for the nonlinear poroelasticity model. Compared with the linear poroelasticiy model, as the nonlinear term brings a lot of troubles in analysis and numerical tests, there exists few numerical methods for the nonlinear poroelasticity model. In [9], the finite element method is used to solve the nonlinear problem. In [31], the authors proposed and analyzed the H(div)-conforming finite element methods for a nonlinear poroelasticity model. In [19], the constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) is used to solve a nonlinear poroelasticity problem. In [30], discontinuous Galerkin method is used to solve the quasi-static nonlinear Biot's model. In [5], a hybrid high-order method is proposed to solve a nonlinear elasticity problem. Some other numerical methods for nonlinear poroelasticity model can also be found in [10, 11]. In this paper, to overcome the locking phenomenon of displacement variable and pressure oscillation, we reformulate the original problem to a new coupling system which consists a generalized nonlinear Stokes problem of displacement vector field with a pseudo pressure and a diffusion problem of other pseudo pressure fields. At the same time, based on the idea of [21] (the authors propose a multirate time iterative scheme based on multiphysics discontinuous Galerkin for a linear poroelsticity model), we make full use of this characteristics that the displacement vector field changes slowly with time while the diffusion problem changes rapidly with time to design a multirate time iterative scheme based on multiphysics finite element method to solve the nonlinear poroelasticity problem – a multiphysics finite element method for the spatial discretization, the generalized nonlinear Stokes problem solved in a coarse time step and the diffusion problem in a finer time step. Then, we give the stability analysis and the convergence analysis for the multirate iterative scheme. The proposed algorithm is a decoupled algorithm, which is easily implemented in computation and reduce greatly computation cost. To the best of our knowledge, it is the first time to propose and analyze a multirate iterative scheme with multiphysics finite element method for the nonlinear poroelasticity model.

The remainder of this paper is organized as follows. In Section 2, we will give the multiphysics reformulation of the nonlinear poroelasticity model. In Section 3, we first give the multirate iterative scheme for the reformed poroelasticity model and then convergence analysis is performed for the multirate iterative scheme. In Section 4, we show some numerical examples to verify that the multirate iterative scheme not only greatly reduces the computational cost, but also has no numerical oscillation. Finally, we draw a conclusion to summarize the main results of this paper.

2. Nonlinear Poroelasticity Model and Multiphysics Reformulation

In this paper, we consider the following quasi-static nonlinear poroelasticity model (cf. [16, 17, 26]):

$$-\mathrm{div}\sigma(u) + \alpha\Delta p = \mathbf{f} \quad \text{in } \Omega_T := \Omega \times (0,T) \subset \mathbf{R}^d \times (0,T), \tag{2.1}$$

$$(c_0 p + \alpha \mathrm{div} u)_t + \mathrm{div} v_f = \phi \quad \text{in } \Omega_T, \tag{2.2}$$

where

$$v_f := -\frac{K}{\mu_f}(\Delta p - \rho_f \mathbf{g}). \tag{2.3}$$

Here Ω is a bounded polygonal domain in $R^d\,(d = 2,3)$ and $\partial\Omega$ is its boundary. $\mathbf{u}$ is the displacement vector of the solid, p is the pressure of the solvent, $\mathbf{f}$ is the body force and I