JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 1, pp. 25-46

doi: 10.4208/jpde.v37.n1.2
March 2024

Robustness of Pullback Attractors for 2D Incompressible Navier-Stokes Equations with Delay

SU Keqin[1] and YANG Xinguang[2,*]

[1] *College of Information and Management Science, Henan Agricultural University, Zhengzhou 450046, China.*
[2] *Department of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China.*

Received 25 May 2022; Accepted 19 April 2023

Abstract. This paper is concerned with the pullback dynamics and robustness for the 2D incompressible Navier-Stokes equations with delay on the convective term in bounded domain. Under appropriate assumption on the delay term, we establish the existence of pullback attractors for the fluid flow model, which is dependent on the past state. Inspired by the idea in Zelati and Gal's paper (JMFM, 2015), the robustness of pullback attractors has been proved via upper semi-continuity in last section.

AMS Subject Classifications: 35B40, 35B41, 35Q30, 76D03, 76D05

Chinese Library Classifications: O175.29

Key Words: Navier-Stokes equations; pullback attractors; delay; upper semi-continuity.

1 Introduction

This presented is concerned with the pullback dynamics and robustness for the 2D Navier-Stokes equations with delay, which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u(t-\rho(t))\cdot\nabla)u + \nabla p = f(t), & (t,x)\in(\tau,+\infty)\times\Omega, \\ \mathrm{div} u = 0, & (t,x)\in(\tau,+\infty)\times\Omega, \\ u(t,x) = 0, & (t,x)\in(\tau,+\infty)\times\partial\Omega, \\ u(\tau+\theta,x) = \phi(\theta), & (\theta,x)\in[-h,0]\times\Omega, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `keqinsu@hotmail.com` (K. Q. Su), `yangxinguang@hotmail.com` (X. G. Yang)

where the kinematic viscosity $\nu>0$, p is the unknown pressure, the external force term is $f(t)$, the delay function $\rho(t)\in C^1(\mathbb{R};[0,h])$ with

$$0<\rho'(t)\leq M<1 \text{ for all } t\in\mathbb{R},$$

$h>0$ is a constant, $u_t(s)=u(t+s)$, $s\in[-h,0]$, $\phi(\theta)$ is the initial datum in $[-h,0]$, and $u(\tau,x)=\phi(0)$.

The research on dynamic systems for the two dimensional incompressible Navier-Stokes equations has attracted mathematician's attention in 1980s, which contains the existence of attractors and its geometric structure on different domains, see [1–3] and the literatures therein. Delay effect can be found in many aspects, such as biology, economic and so on, which can lead to the instability of system, even if the delay is very small. Since the motion of fluid flow is not only dependent on current state, but also the past history such as delay and memory, which leads to the research on incompressible functional Navier-Stokes equations and some extended models. In 1963, Krasovskii [4] first noticed the system with delay, constructed the Navier-Stokes equations with delay and obtained the well-posedness of system. In past decades, there were many literatures about the hydrodynamic system with delay, especially the Navier-Stokes equations with constant, variable and distributed delays, which can be referred to, Taniguchi [5], Hale [6], Caraballo and Real [7], Garcín-Luengo, Marín-Rubio and Planas [8–10] and more literatures therein about the fluid flow with delays.

The hydrodynamic system with perturbation is a key research point in last thirty decades, which includes the convergence of attractors as perturbation vanishes, i.e., the robustness of attractors via upper and lower semi-continuity, see the theory and application in Chapter III of Carvalho, Langa and Robinson [11]. However, the lower semi-continuity is very difficult to verify since the lack of good regularity, which leads to the validity of upper semi-continuity for attractors as a tool to understand turbulence, see [11–13]. In 2009, another interesting method was given by Wang [14] to obtain the upper semi-continuity of random attractors, and Wang [15] for the pullback attractors. For more relating results to the convergence of attractors and solutions, we can refer to [16–18] and so on.

To our best knowledge, there are fruitful results on the dynamics for Navier-Stokes equations and related models with delay, which illustrated the complexity of fluid flow. However, the research on robustness of attractors for incompressible Navier-Stokes equations with delay on convective term, i.e., the convergence of pullback attractors as delay vanishes is still open, which is our main goal in this paper. The main features of this paper can be summarized as follows:

(1) In Section 2, some functional spaces and related conclusions on pullback $\mathcal{D}$-attractors are given. The definition of upper semicontinuity of attractors for system with delay term is presented, which is inspired by [17];

(2) In Section 3, we use the standard Galerkin method and conclusions on compactness to derive the wellposedness of solutions, and determine a continuous semi-flow