Numer. Math. Theor. Meth. Appl. Vol. **17**, No. 1, pp. 69-92
doi: 10.4208/nmtma.OA-2023-0065 February 2024

On Polar Decomposition of Tensors with Einstein Product and a Novel Iterative Parametric Method

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Received 17 May 2023; Accepted (in revised version) 9 September 2023

Abstract. This study aims to investigate the polar decomposition of tensors with the Einstein product for the first time. The polar decomposition of tensors can be computed using the singular value decomposition of the tensors with the Einstein product. In the following, some iterative methods for finding the polar decomposition of matrices have been developed into iterative methods to compute the polar decomposition of tensors. Then, we propose a novel parametric iterative method to find the polar decomposition of tensors. Under the obtained conditions, we prove that the proposed parametric method has the order of convergence four. In every iteration of the proposed method, only four Einstein products are required, while other iterative methods need to calculate multiple Einstein products and one tensor inversion in each iteration. Thus, the new method is superior in terms of efficiency index. Finally, the numerical comparisons performed among several well-known methods, show that the proposed method is remarkably efficient and accurate.

AMS subject classifications: 15A24, 65F10, 65F30

Key words: Iterative methods, Einstein product, polar decomposition of a tensor, polar factor, order of convergence.

1. Introduction

Throughout this study, matrices are denoted by uppercase letters $A, B, \dots,$ and tensors are written in calligraphic font $\mathcal{A}, \mathcal{B}, \dots$. Tensors occur in a wide variety of application areas such as in document analysis, psychometrics, formulation an n-person noncooperative game, medical engineering, chemometrics, higher-order, and so on [6, 15, 18, 24, 25]. Suppose that N is a positive integer, an N-th order tensor $\mathcal{A} = (a_{i_1\dots i_N})_{1\le i_j\le P_j}$ is a multidimensional array with $P_1 \dots P_N$ entries. The tensor $\mathcal{A}$ is

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called a hyper-matrix or the tensor is the higher-order generalization of vectors and matrices.

In the following, we give some definitions of tensors and the Einstein product which are used in the body of this manuscript [3,4,12,13,20].

Definition 1.1 ([5]). *Let N and M be positive integers, also $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ and $\mathcal{B} \in \mathbb{R}^{Q_1\times\cdots\times Q_N\times K_1\times\cdots\times K_M}$ are tensors. Then the Einstein product of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:*

$$(\mathcal{A} *_N \mathcal{B})_{p_1\ldots p_N k_1\ldots k_M} := \sum_{q_N}^{Q_N}\cdots\sum_{q_1}^{Q_1} a_{p_1\ldots p_N q_1\ldots q_N} b_{q_1\ldots q_N k_1\ldots k_M}, \tag{1.1}$$

therefore, $\mathcal{A} *_N \mathcal{B} \in \mathbb{R}^{P_1\times\cdots\times P_N\times K_1\times\cdots\times K_M}$.

Note that if $N = M = 1$, the Einstein product reduces to the standard matrix multiplication.

Definition 1.2 ([5]). *Let $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ be a tensor, then transpose and Frobenius norm of the tensor $\mathcal{A}$ are defined as follows:*

$$(\mathcal{A}^T)_{p_1\ldots p_N q_1\ldots q_N} := (\mathcal{A})_{q_1\ldots q_N p_1\ldots p_N},$$

and

$$\|\mathcal{A}\| := \sqrt{\sum_{q_1\ldots q_N, p_1\ldots p_N} (a_{q_1\ldots q_N p_1\ldots p_N})^2},$$

respectively.

Definition 1.3. *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is a symmetric tensor if $\mathcal{A} = \mathcal{A}^T$, it means*

$$a_{p_1,p_2,\ldots,p_n,q_1,q_2,\ldots,q_n} = a_{q_1,q_2,\ldots,q_n,p_1,p_2,\ldots,p_n}.$$

Definition 1.4 ([5]). *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is said to be a diagonal tensor if $a_{p_1\ldots p_N q_1\ldots q_N} = 0$ for $p_l \neq q_l$, $l = 1,\ldots,N$. A diagonal tensor $\mathcal{I} \in \mathbb{R}^{P_1\times\cdots\times P_N\times P_1\times\cdots\times P_N}$ is identity if*

$$i_{p_1\ldots p_N q_1\ldots q_N} = \Pi_{k=1}^{N} \delta_{p_l q_l},$$

where

$$\delta_{p_l q_l} = \begin{cases} 1, & p_l = q_l, \\ 0, & p_l \neq q_l. \end{cases}$$

Definition 1.5. *A tensor $\mathcal{A} \in \mathbb{R}^{P_1\times\cdots\times P_N\times Q_1\times\cdots\times Q_N}$ is an orthogonal tensor if*

$$\mathcal{A}^T *_N \mathcal{A} = \mathcal{I}.$$