Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2024-0003

Vol. **3**, No. 1, pp. 61-120
February 2024

Time-Velocity Decay of Solutions to the Non-cutoff Boltzmann Equation in the Whole Space

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Received 29 January 2024; Accepted 14 February 2024

Abstract. In this paper, we consider the perturbed solutions with polynomial tail in large velocities for the non-cutoff Boltzmann equation near global Maxwellians in the whole space. The global in time existence is proved in the weighted Sobolev spaces and the almost optimal time decay is obtained in Fourier transform based low-regularity spaces. The result shows a time-velocity decay structure of solutions that can be decomposed into two parts. One part allows the slow polynomial tail in large velocities, carries the initial data and enjoys the exponential or arbitrarily large polynomial time decay. The other part, with zero initial data, is dominated by the non-negative definite symmetric dissipation and has the exponential velocity decay but only the slow polynomial time decay.

AMS subject classifications: 35Q20, 35B35

Key words: Boltzmann equation, angular non-cutoff, large time behavior.

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1 Introduction

We consider the following Cauchy problem on the spatially inhomogeneous non-cutoff Boltzmann equation in the whole space:

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$$\partial_t F + v\cdot\nabla_x F = Q(F,F), \quad F(0,x,v) = F_0(x,v), \tag{1.1}$$

where the unknown $F(t,x,v)\geq 0$ stands for the density distribution function of rarefied gas particles with velocity $v\in\mathbb{R}^3$ at position $x\in\mathbb{R}^3$ and time $t>0$, and initial data $F_0(x,v)\geq 0$ is given. The bilinear Boltzmann collision operator $Q(\cdot,\cdot)$ which acts only on velocity variable is defined by

$$Q(G,F)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\sigma)[G(u')F(v') - G(u)F(v)]\,d\sigma du,$$

where the post-collision velocities (v',u') denote

$$v' = \frac{v+u}{2} + \frac{|v-u|}{2}\sigma, \quad u' = \frac{v+u}{2} - \frac{|v-u|}{2}\sigma, \quad \sigma\in\mathbb{S}^2.$$

Moreover, we assume that the non-negative Boltzmann collision kernel $B(v-u,\sigma)$ takes the form

$$B(v-u,\sigma) = |v-u|^\gamma b(\cos\theta), \quad -3<\gamma\leq 1,$$

where

$$\cos\theta = \frac{v-u}{|v-u|}\cdot\sigma, \quad 0<\theta\leq\frac{\pi}{2},$$

and

$$\sin\theta b(\cos\theta)\sim\theta^{-1-2s} \quad \text{as} \quad \theta\to 0, \quad 0<s<1. \tag{1.2}$$

Define the global Maxwellian μ by

$$\mu = \mu(v) := (2\pi)^{-3/2} e^{-|v|^2/2}.$$

Under the perturbation near the global Maxwellian, we look for solutions in the form of

$$F = \mu + g \tag{1.3}$$

for the unknown function $g = g(t,x,v)$. Substituting (1.3) into (1.1), we can rewrite the Cauchy problem on the Boltzmann equation in terms of g as

$$\begin{aligned} &\partial_t g + v\cdot\nabla_x g = \mathcal{L}g + Q(g,g), \\ &g(0,x,v) = g_0(x,v) := F_0(x,v) - \mu(v), \end{aligned} \tag{1.4}$$

where the linearized collision operator $\mathcal{L}$ is given by

$$\mathcal{L}g := Q(\mu,g) + Q(g,\mu).$$