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The Boundedness Below of 2×2 Upper Triangular Linear Relation Matrices

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Abstract. In this note, the boundedness below of linear relation matrix $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in LR(H\oplus K)$ is considered, where $A \in CLR(H)$, $B \in CLR(K)$,$C \in BLR(K,H)$, H,K are separable Hilbert spaces. By suitable space decompositions, a necessary and sufficient condition for diagonal relations A,B is given so that M_C is bounded below for some $C \in BLR(K,H)$. Besides, the characterization of $\sigma_{ap}(M_C)$ and $\sigma_{su}(M_C)$ are obtained, and the relationship between $\sigma_{ap}(M_0)$ and $\sigma_{ap}(M_C)$ is further presented.

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Key words: Linear relation matrix, boundedness below, approximate point spectrum, space decomposition.

1 Introduction

The linear relation established by Neumann [1] was first introduced into functional analysis, whose motivation is to consider the adjoint of linear differential operators that are not densely defined. It has extensive applications in nonlinear analysis, differential equations, and optimization and control problems. For instance, the port-Hamiltonian formulation can be conveniently established by the linear relation language, in which the kernel of certain row relations (dually, the range of column relations) and the structure of the involved port-Hamiltonian pencils play significant roles [2]. The simplest naturally occurring example of a linear relation is the inverse of a linear mapping $A:X\to Y$, defined by the set of solutions

$$A^{-1}y:=\{x\in X: Ax=y\}$$

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of the equation $Ax=y$. Arens [3] studied the resolvent set and the spectrum of linear relations, and obtained the existence theorem of self-adjoint relations.

Suppose H and K are separable Hilbert spaces with infinite dimension. A linear relation $A:H\to K$ is a mapping from the subspace

$$\mathcal{D}(A)=\{x\in H: Ax\neq\emptyset\}\subset H,$$

called the domain of A, into the set of non-empty subsets of K, and for all non-zero scalars c_1,c_2 and $x_1,x_2\in\mathcal{D}(A)$ such that

$$A(c_1x_1+c_2x_2)=c_1Ax_1+c_2Ax_2.$$

We introduce $LR(H,K)$ to represent the class of linear relation from H into K and write $LR(H)=LR(H,H)$. The graph of A is defined by

$$G(A)=\{(x,y)\in H\oplus K: x\in\mathcal{D}(A), y\in Ax\}.$$

Like operator case, A is called a closed linear relation provided that $G(A)$ is closed. The collection of all of closed linear relations is represented by $CLR(H,K)$. The relation A^{-1} is determined by

$$G(A^{-1})=\{(y,x)\in K\oplus H: (x,y)\in G(A)\}.$$

The range of A is designed by $\operatorname{ran}(A)=A(\mathcal{D}(A))$, and the kernel by $\ker(A)=\{x\in H: (x,0)\in G(A)\}$; write $\alpha(A)=\dim\ker(A)$ and $\beta(A)=\operatorname{codim}\operatorname{ran}(A)$. If $\operatorname{ran}(A)=K$ ($\ker(A)=\{0\}$), A is called surjective (injective). Note that for $x\in\mathcal{D}(A)$,

$$y\in Ax\Leftrightarrow Ax=y+A(0).$$

If A is a linear relation from H into K, then we use Q_A to represent the quotient mapping $Q^K_{\overline{A(0)}}\in L(K,K/\overline{A(0)})$, and hence Q_AA is obviously an operator. For $x\in\mathcal{D}(A)$,

$$Q_AAx=Q_Ay,\quad \text{for all } y\in Ax.$$

For $x\in\mathcal{D}(A)$, we define

$$||Ax||=||Q_AAx||.$$

The norm of A is denoted by $||A||=||Q_AA||$. This quantity is semi-norm, because $||A||=0$ does not imply $A=0$. If $||A||<+\infty$, then it is said that A is bounded. The set of all the bounded linear relations defined everywhere is represented by $BLR(H,K)$. The resolvent set of a linear relation $A\in LR(H)$ can be expressed as

$$\rho(A)=\{\lambda\in\mathbb{C}:\ (A-\lambda)^{-1} \text{ is bounded and single valued}\},$$

and the spectrum is defined by $\sigma(A)=\mathbb{C}\backslash\rho(A)$. Recall that $A\in LR(H,K)$ is called bounded below if there exists $\delta>0$ such that $||Ax||\geq\delta||x||$ for each $x\in\mathcal{D}(A)$. The approximate point spectrum $\sigma_{ap}(A)$ and the surjective spectrum $\sigma_{su}(A)$ are defined respectively by

$$\sigma_{ap}(A)=\{\lambda\in\mathbb{C}: A-\lambda \text{ is not bounded below}\},$$