J. Math. Study
doi: 10.4208/jms.v57n1.24.05

Vol. **57**, No. 1, pp. 84-100
March 2024

Non-Regular Pseudo-Differential Operators on Matrix Weighted Besov-Triebel-Lizorkin Spaces

Tengfei Bai[1] and Jingshi Xu[2,*]

[1] *School of Mathematics and Statistics, Hainan Normal University, Haikou, Hainan 571158, China;*
[2] *Center for Applied Mathematics of Guangxi, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 54100, China.*

Received April 5, 2023; Accepted October 16, 2023;
Published online March 21, 2024.

Abstract. In this paper we obtain the boundedness of non-regular pseudo-differential operators with symbols in Besov spaces on matrix-weighted Besov-Triebel-Lizorkin spaces. These symbols include the classical Hörmander classes.

AMS subject classifications: 35S05, 46E30

Key words: Pseudo-differential operator, matrix weight, Besov space, Triebel-Lizorkin space.

1 Introduction

The pseudo-differential operators have been widely used in plenty of mathematical areas; see [1, 11, 14, 16, 17, 28, 31–33, 36, 38]. The boundedness of pseudo-differential operators on Triebel-Lizorkin and Besov spaces has been considered in [6, 22–25, 27, 30]. The authors of the paper proved the boundedness of the Hörmander classes pseudo-differential operators on matrix-weighted Besov spaces and Triebel-Lizorkin spaces in [2].

In [18, 21], Marschall obtained the boundedness of non-regular pseudo-differential operators corresponding to symbols in the class $SB^m_\delta(r,\mu,\nu;N,\lambda)$ (see Definition 2.5) on Triebel-Lizorkin spaces and Besov spaces. Then Sato obtained the boundedness of non-regular pseudo-differential operators on the weighted Triebel-Lizorkin spaces in [29], and Drihem and Hebbache obtained the boundedness of non-regular pseudodifferential operators on variable Triebel-Lizorkin spaces in [7].

In the last three decades, inspired by the applications of matrix-weighted functions, many matrix-weighted function spaces have appeared, such as matrix-weighted Lebesgue

*Corresponding author. *Email addresses:* `tfeibai@qq.com` (Bai T), `jingshixu@126.com` (Xu J)

spaces [13,35], matrix weighted Besov and Triebel-Lizorkin spaces [4,9,10,26,37], matrix-weighted Besov type spaces and Triebel-Lizorkin type spaces [3]. In [37], Wang, Yang and Zhang obtained the characterizations of matrix-weighted Triebel-Lizorkin spaces in terms of the Peetre maximal function, the Lusin area function, and the Littlewood-Paley g_λ^*-function. They also proved the boundedness of Fourier multipliers on matrix-weighted Triebel-Lizorkin spaces under the generalized Hörmander condition. In [3], Bu, Hytönen, Yang, Yuan proposed a new concept of A_p-dimension of matrix weights. Then they obtained the boundedness of φ-transform, pseudo-differential operators, trace operators, and Calderón-Zygmund operators on matrix-weighted Besov type spaces and Triebel-Lizorkin type spaces. In particular, the symbols of their pseudo-differential operators are in the classical Hörmander class $S^m_{1,1}$.

Since the class $SB^m_\delta(r,\mu,\nu;N,\lambda)$ includes some Hörmander classes as special cases, in this paper, we consider the boundedness of non-regular pseudo-differential operators with symbols in $SB^m_\delta(r,\mu,\nu;N,\lambda)$ on matrix-weighted Besov spaces and Triebel-Lizorkin spaces.

This paper is organized as follows. In Section 2, we give some convenient notations and recall several concepts about matrix weights and function spaces. Some key lemmas and basic tools are given in Section 3. The boundedness of non-regular pseudo-differential operators on matrix-weighted Besov spaces and Triebel-Lizorkin spaces are described in Section 4.

2 Preliminaries

Let χ_E be the characteristic function of the set $E\subset\mathbb{R}^n$. Let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The Fourier transform of f is defined by $\mathcal{F}(f):=\hat{f}:=\int_{\mathbb{R}^n}f(x)e^{-2\pi ix\cdot\xi}\mathrm{d}x$ and the inverse Fourier transform of f by $\mathcal{F}^{-1}(f):=\check{f}:=\int_{\mathbb{R}^n}f(x)e^{2\pi ix\cdot\xi}\mathrm{d}x$. Let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space, and let $\mathscr{S}'(\mathbb{R}^n)$ be its dual.

Definition 2.1. *Let φ_0 be a Schwartz function such that supp$(\varphi_0)\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$ and $\varphi_0(\xi)=1$ for $|\xi|\le 1$. Moreover, put $\varphi_j(\xi)=\varphi_0(2^{-j}\xi)-\varphi_0(2^{-j+1}\xi)$ for $j\in\mathbb{N}$. Then supp$(\varphi_j)\subset\{\xi:2^{j-1}\le|\xi|\le 2^{j+1}\}$ for all $j\in\mathbb{N}$ and*

$$\sum_{j=0}^{\infty}\varphi_j(\xi)=1$$

for $\xi\in\mathbb{R}^n$. Hence $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity on $\mathbb{R}^n$ subordinated to the dyadic rings $\{\xi:2^{j-1}\le|\xi|\le 2^{j+1}\}$, $j\in\mathbb{N}$, and $\overline{B(0,2)}$.

We also set $\tilde{\varphi}_0:=\varphi_0+\varphi_1$, and $\tilde{\varphi}_j:=\varphi_{j-1}+\varphi_j+\varphi_{j+1}$ for $j\in\mathbb{N}$. Note that, $\varphi_j\tilde{\varphi}_j=\varphi_j$ for $j\in\mathbb{N}_0$ and

$$\begin{aligned}
&\operatorname{supp}(\tilde{\varphi}_j)\subset\{\xi\in\mathbb{R}^n:2^{j-2}\le|\xi|\le 2^{j+2}\} &&\text{for } j\ge 2,\\
&\operatorname{supp}(\tilde{\varphi}_j)\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2^{j+2}\} &&\text{for } j=0,1.
\end{aligned}$$