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# WEAK APPROXIMATIONS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH FRACTIONAL NOISE*

Meng Cai, Siqing Gan[1) and Xiaojie Wang
*School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha, China*
*Emails: csumathcai@csu.edu.cn, sqgan@csu.edu.cn;*
*x.j.wang7@csu.edu.cn, x.j.wang7@gmail.com*

**Abstract**

This paper aims to analyze the weak approximation error of a fully discrete scheme for a class of semi-linear parabolic stochastic partial differential equations (SPDEs) driven by additive fractional Brownian motions with the Hurst parameter $H \in (1/2, 1)$. The spatial approximation is performed by a spectral Galerkin method and the temporal discretization by an exponential Euler method. As far as we know, the weak error analysis for approximations of fractional noise driven SPDEs is absent in the literature. A key difficulty in the analysis is caused by the lack of the associated Kolmogorov equations. In the present work, a novel and efficient approach is presented to carry out the weak error analysis for the approximations, which does not rely on the associated Kolmogorov equations but relies on the Malliavin calculus. To the best of our knowledge, the rates of weak convergence, shown to be higher than the strong convergence rates, are revealed in the fractional noise driven SPDE setting for the first time. Numerical examples corroborate the claimed weak orders of convergence.

*Mathematics subject classification:* 60H35, 60H15, 65C30.
*Key words:* Parabolic SPDEs, Fractional Brownian motion, Weak convergence rates, Spectral Galerkin method, Exponential Euler method, Malliavin calculus.

## 1. Introduction

As an extension of the classical Brownian motion ($H = 1/2$), the fractional Brownian motion (fBm in short) with Hurst parameter $H \in (0, 1)$ has rapidly become an extremely hot topic, impacting on a wide range of application areas such as hydrology, financial markets, telecommunications, medicine and so on (see, e.g., [10,12,16,17,19,22,23] and references therein). It is therefore natural and meaningful to investigate some physical phenomena with randomness modeled by fBm driven SPDEs from both theoretical and numerical points of view. In contrast to the extensive researches for SPDEs driven by standard $Q$-Wiener process (see monographs [20, 21] for numerous references), the study of the SPDEs driven by fBm is in its beginning. Recently, there has been a fast increasing number of works on theoretical analysis of SPDEs driven by fBm e.g., [10, 11, 15, 23, 24], however, finding reliable numerical approximations is still an active ongoing research area. For instance, Cao, Hong and Liu [5, 6] get the optimal strong error analysis for SPDEs with additive noise which is fractional in space and white in time. With the Hurst parameter $H \in (1/2, 1)$, the authors in [27] derive sharp mean-square regularity results and optimal strong convergence rates for the linear implicit fully discrete

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1) Corresponding author

method. Besides, an explicit method has been analyzed in [13] and the super-convergence rate has been reached for SPDEs with spatially smooth fractional noise. As far as we know, the corresponding weak error analysis for numerical approximations of SPDEs with fractional noise is absent in existing literature.

In this work, we attempt to fill the gap by analyzing convergence rates of weak approximations for the following stochastic parabolic equation driven by a cylindrical fractional noise:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = F(X(t))\,\mathrm{d}t + \Psi \mathrm{d}W^H(t), & t \in (0,T], \\ X(0) = \xi. \end{cases} \tag{1.1}$$

Here, given a real separable Hilbert space $(U, \langle\cdot,\cdot\rangle, \|\cdot\|)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $A\colon \mathrm{Dom}(A) \subset U \to U$ be a densely defined, linear unbounded, positive self-adjoint operator with compact inverse and let $F\colon U \to U$ and $\Psi\colon U \to U$ be deterministic mappings. By adopting the approach used in [11], we define the standard cylindrical fBm $\{W^H(t)\}_{t\in[0,T]}$ with Hurst parameter $H \in (1/2, 1)$ as the following formal series:

$$W^H(t) := \sum_{n=1}^{\infty} w_n^H(t) e_n, \quad t \in [0,T], \tag{1.2}$$

where $\{w_n^H(t)\}_{n\in\mathbb{N}}, t \in [0,T]$ is a sequence of independent real-valued standard fBm each with the same Hurst parameter $H \in (1/2, 1)$ and $\{e_n\}_{n\in\mathbb{N}}$ is a complete orthonormal basis of $U$. Denoted by $E(t) = \exp(-tA), t \geq 0$ the analytic semigroup generated by $-A$, under certain assumptions specified later, (1.1) admits a unique mild solution $X : [0,T] \times \Omega \to U$, given by

$$X(t) = E(t)\xi + \int_0^t E(t-s)F(X(s))\,\mathrm{d}s + \int_0^t E(t-s)\Psi \mathrm{d}W^H(s), \quad \mathbb{P}\text{-a.s.}, \tag{1.3}$$

where the stochastic integral is defined in [10]. Note that the mild solution (1.3) is rarely known explicitly and therefore numerical approximations are often helpful. To rigorously justify the use of the classical Malliavin calculus, firstly, we truncate $W^H(t)$ as

$$W_K^H(t) := \sum_{n=1}^{K} w_n^H(t) e_n, \quad t \in [0,T], \tag{1.4}$$

and numerically solve the following SPDE driven by the $K$-dimensional fBm

$$\begin{cases} \mathrm{d}X^K(t) + AX^K(t)\mathrm{d}t = F(X^K(t))\mathrm{d}t + \Psi \mathrm{d}W_K^H(t), & t \in (0,T], \\ X^K(0) = \xi. \end{cases} \tag{1.5}$$

The full discretization of (1.5) is realized by combining the spectral Galerkin method in space with the exponential Euler method in time. To be specific, the spatial spectral Galerkin method is given by

$$\begin{cases} \mathrm{d}X^{K,N}(t) + A_N X^{K,N}(t)\mathrm{d}t = P_N F(X^{K,N}(t))\mathrm{d}t + P_N \Psi \mathrm{d}W_K^H(t), & t \in (0,T], \\ X^{K,N}(0) = P_N \xi, \end{cases} \tag{1.6}$$

and the temporal exponential Euler method reads

$$\begin{aligned} X_{t_{m+1}}^{K,M,N} &= E_N(\tau) X_{t_m}^{K,M,N} + \tau E_N(\tau) F(X_{t_m}^{K,M,N}) \\ &\quad + E_N(\tau)\Psi \Delta W_K^H(m), \quad m = 0, 1, \ldots, M-1. \end{aligned} \tag{1.7}$$