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A LOW-COST OPTIMIZATION APPROACH FOR SOLVING MINIMUM NORM LINEAR SYSTEMS AND LINEAR LEAST-SQUARES PROBLEMS*

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Abstract

Recently, the authors proposed a low-cost approach, named Optimization Approach for Linear Systems (OPALS) for solving any kind of a consistent linear system regarding the structure, characteristics, and dimension of the coefficient matrix A. The results obtained by this approach for matrices with no structure and with indefinite symmetric part were encouraging when compare with other recent and well-known techniques. In this work, we proposed to extend the OPALS approach for solving the Linear Least-Squares Problem (LLSP) and the Minimum Norm Linear System Problem (MNLSP) using any iterative low-cost gradient-type method, avoiding the construction of the matrices A^TA or AA^T, and taking full advantage of the structure and form of the gradient of the proposed nonlinear objective function in the gradient direction. The combination of those conditions together with the choice of the initial iterate allow us to produce a novel and efficient low-cost numerical scheme for solving both problems. Moreover, the scheme presented in this work can also be used and extended for the weighted minimum norm linear systems and minimum norm linear least-squares problems. We include encouraging numerical results to illustrate the practical behavior of the proposed schemes.

Mathematics subject classification: 65F10, 65F20, 90C06.
Key words: Nonlinear convex optimization, Gradient-type methods, Spectral gradient method, Minimum norm solution linear systems, Linear least-squares solution.

1. Introduction

The adjustment of a data set by linear mathematical models, that yield systems of the form $Ax = b$, appears in a large number of areas, from the scientific field to the social field, since in many situations a linear behavior is found to be an effective simple approximation to the problem. However, frequently this linear model involves an over-determined linear system of equations with no solutions. In this case, the adjustment is made by solving the minimum square norm of the linear system, which is known as a Linear Least-Squares Problem (LLSP)

$$\min_{x\in R^n} \frac{1}{2}\|Ax-b\|_2^2, \tag{1.1}$$

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where $A \in R^{m\times n}$ and $b \in R^m$. If, on the other hand, the adjustment of the data set to a linear model results in an under-determined system with more than one solution, the most suitable solution corresponds to the minimum norm solution of the linear system of equations, known as the Minimum Norm Linear System Problem (MNLSP)

$$\begin{cases} \min_{x\in R^n} \dfrac{1}{2}\|x\|_2^2, \\ \text{subject to} \quad Ax = b. \end{cases} \tag{1.2}$$

Furthermore, in some applications, as for example the location of sources in magnetoencephalography, the solution of problem (1.2) favors the sources closest to the surface, so in this case it is necessary to add weights to the location of the different sources, and then a weighted matrix W must be considered, see e.g., [15, 17, 19]. The incorporation of the matrix W in problem (1.2) is a generalization of this problem that will be denote as Weighted Minimum Norm Linear System Problem (WMNLSP). This generalization can be written as

$$\begin{cases} \min_{x\in R^n} \dfrac{1}{2}\|Wx\|_2^2, \\ \text{subject to} \quad Ax = b. \end{cases} \tag{1.3}$$

Clearly, problem (1.2) is a particular case of the WMNLSP (1.3), where the matrix $W = I_n$.

In some other applications, where the LLSP has infinite solutions, a minimum norm solution of a linear least-squares problem is the desired solution. Then a Minimum Norm Linear Least-Squares Problem (MNLLSP), where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, needs to be solved,

$$\begin{cases} \min_{x\in \mathbb{R}^n} \dfrac{1}{2}\|x\|_2^2, \\ \text{subject to} \quad \|Ax - b\|_2^2. \end{cases} \tag{1.4}$$

Problems (1.1)-(1.4) are different. The first one is a quadratic unconstrained optimization problem and the others are quadratic constrained optimization problems, where the restrictions are the linear equations of an under-determined system, except for problem (1.4). In the literature there are different methods for solving the LLSP as for example the Normal Equations System (NES), $A^TAx = A^Tb$, the QR factorization of the matrix A, the SVD factorization of the matrix A, see e.g., [6, 16, 22]. The major disadvantage of solving the NES is the increase in the value of the condition number of the normal equation matrix A^TA, which introduces rounding errors. On the other hand, the classical solution of the MNLSP, consists on solving the second-order normal equations, $AA^Ty = b$ and then the solution of the problem is obtained as $x = A^Ty$. This last scheme uses $\mathcal{O}(m^2n/2 + m^3/6)$ floating point operations and has the disadvantage that the formation of the matrix AA^T introduces numerical errors in the approach. Also, this problem can be solved using the QR factorization of the matrix associated with the original system. Moreover, the QR decomposition can be done through the Gram-Schmidt orthogonalization method, whose modified version uses $\mathcal{O}(m^2n)$ floating point operations or through the application of Householder reflections that requires $\mathcal{O}(m^2n - m^3/3)$ floating point operations, see e.g., [6, 16, 22]. As a consequence, it is of great interest to propose low-cost and efficient numerical schemes for solving large scale LLSP and MNLSP, for which the normal equation matrices do not need to be built and few requirements on the matrix associated to the linear system are needed.

In [7], the authors proposed a non-linear approach, named OPALS (Optimization Approach for Linear Systems), for solving any kind of consistent linear systems regarding the structure,