Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0207

Vol. **35**, No. 3, pp. 553-578
March 2024

Solution of Inverse Geometric Problems Using a Non-Iterative MFS

Andreas Karageorghis[1,*], Daniel Lesnic[2] and Liviu Marin[3]

[1] *Department of Mathematics and Statistics, University of Cyprus/* Πανεπιστήμιο Κύπρου*, P.O. Box 20537, 1678 Nicosia/*Λευκωσία*, Cyprus/*Κύπρος*.*
[2] *Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK.*
[3] *Department of Mathematics, Faculty of Mathematics and Computer Science, University of Bucharest, 14 Academiei, 010014 Bucharest, and "Gheorghe Mihoc – Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 13 Calea 13 Septembrie, 050711 Bucharest, Romania.*

Received 3 August 2023; Accepted (in revised version) 3 January 2024

Abstract. In most of the method of fundamental solutions (MFS) approaches employed so far for the solution of inverse geometric problems, the MFS implementation typically leads to non–linear systems which were solved by standard nonlinear iterative least squares software. In the current approach, we apply a three–step non–iterative MFS technique for identifying a rigid inclusion from internal data measurements, which consists of: (i) a direct problem to calculate the solution at the set of measurement points, (ii) the solution of an ill–posed linear problem to determine the solution on a known virtual boundary and (iii) the solution of a direct problem in the virtual domain which leads to the identification of the unknown curve using the MATLAB® functions `contour` in 2D and `isosurface` in 3D. The results of several numerical experiments for steady–state heat conduction and linear elasticity in two and three dimensions are presented and analyzed.

AMS subject classifications: 65N35, 65N21, 65N38

Key words: Void detection, inverse problem, method of fundamental solutions.

1 Introduction

The method of fundamental solutions (MFS) has, in recent years, been used extensively for the solution of inverse geometric problems [7, 15]. This is because its main features

*Corresponding author. *Email addresses:* `andreask@ucy.ac.cy` (A. Karageorghis), `amt5ld@maths.leeds.ac.uk` (D. Lesnic), `liviu.marin@fmi.unibuc.ro`; `marin.liviu@gmail.com` (L. Marin)

(meshlessness and ease of application to problems in complex geometries in 2D and 3D) make it ideally suited for the solution of such problems in which the main dependent variable along with part of the boundary are unknown and have to be determined from some extra suitable measurements. The MFS formulations put forward in the literature typically lead to the solution of systems of nonlinear equations which require the use of standard nonlinear iterative software such as the MATLAB® routine `lsqnonlin`, the application of which can be quite costly. For the approach to become less costly, the Jacobian of the system needs to be calculated and provided which is rather tedious [19].

The objective of this paper is to extend the application of the MFS combined with a non-iterative level-set method for identifying a rigid inclusion from internal measurements in the fields of steady-state heat conduction and linear elasticity. Therefore, we shall consider inverse geometric problems of the type considered in [8, 25, 26] using the finite difference method, in [28] using the finite element method and [9, 27] using the boundary element method (BEM). In particular, we shall follow closely the ideas and strategy developed in [27] which are based on the virtual area/volume concept, and adjust them to the MFS. The proposed method leads to the solution of an ill–conditioned system of linear equations to calculate the solution on the virtual boundary. This involves the pseudo–inverse of an ill–conditioned matrix which is calculated and regularized using the truncated singular value decomposition (SVD). We assume that the unknown rigid inclusion is star–shaped and its boundary lies in a known annular domain in between the virtual boundary and a curve on which the internal measurements are situated. Then, the unknown boundary is subsequently recovered by searching for a specific iso–curve in two dimensions (2D) or a specific iso–surface in three dimensions (3D). These searches can be easily carried out utilizing the MATLAB® functions `contour` in 2D and `isosurface` in 3D. We should mention that the proposed method has some similarities to the Kirsch and Kress method [20, 21] proposed by [1], see also [2], for the solution of a different type of inverse geometric problems. This leads to another way of determining the solution on the virtual boundary and the Tikhonov regularization [12, Section 4.4] is applied instead of the truncated SVD.

The type of problems to be solved is presented in Section 2. In Section 3 we describe in detail the application of the method in the three steps which lead to the determination of the unknown boundary. The results of several numerical experiments in 2D and 3D steady–state heat conduction and linear elasticity are analyzed in Sections 4 and 5. Finally, concluding remarks are given in Section 6.

2 Mathematical formulation

We consider the inverse boundary value problem (BVP) for the Laplace equation in $\mathbb{R}^2$ or $\mathbb{R}^3$ [27]

$$\Delta u = 0 \quad \text{in} \quad \Omega := \Omega_2 \backslash \overline{\Omega}_1, \tag{2.1a}$$