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Advection-Pressure Splitting Schemes for the Equations of Blood Flow. Conservative and Non-Conservative Forms

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Abstract. We present a class of simple advection-pressure splitting numerical methods to solve the blood flow equations in compliant arterial vessels. The schemes are inspired by the TV flux vector splitting approach for conservative systems, proposed by Toro and Vázquez [30]. But the reformulated TV-type splitting schemes of this paper have a wider range of applicability, including systems of equations in non-conservative form. The spatial differential operator is split into advection terms, which may be in conservative form, from pressure terms in conservative or non-conservative form. Additionally, unlike the original TV scheme, the reformulated splitting of this paper fully preserves the continuity equation as part of the pressure system. This last feature is consistent with zero-dimensional models for blood flow that are based on neglecting the inertial term in the momentum equation. The schemes are also well suited for systems in which geometric and biomechanical parameters of the problem vary discontinuously. The splitting schemes of this paper are systematically assessed on a carefully designed suite of test problems and compared with several existing, mainstream methods. Overall, the proposed numerical methods perform very satisfactorily and suggest themselves as attractive computational tools for modelling the dynamics of bodily fluids under realistic conditions.

AMS subject classifications: 65M08, 76Z05

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1. Introduction

This paper is concerned with a class of simple numerical methods for solving a system of hyperbolic partial differential equations (PDEs) that govern the fluid dynamics of

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blood in compliant arterial vessels. The one-dimensional PDEs of interest depend on time t and distance x along the vessel and, despite its apparent simplicity, they account for the dynamics of blood inside the vessel, coupled to the simultaneous movement of the vessel wall. Blood vessel walls are not rigid, fixed boundaries. In other words, the alluded equations constitute a simplified fluid-structure interaction (FSI) model. Single vessel segments can be coupled to multiple segments through appropriate coupling conditions so as to form networks of blood vessels that may realistically emulate the circulatory system of mammals. Designing useful numerical methods to solve the equations may pose some challenges. To start with, the hyperbolic character of the equations admits solutions with large spatial and temporal gradients of the unknowns, including shocks, or more specifically, elastic jumps. Moreover, the PDEs include geometrical and biomechanical parameters, which in turn depend on distance along the vessel. Apart from the difficulty of determining such parameters, these may exhibit large spatial gradients, including discontinuities. Such geometrical and biomechanical parameters enter the equations in the form of algebraic source terms and add new features to the PDEs, such as stationary contact discontinuities. Such features constitute additional challenges to the algorithm designer, and the well-balanced concept enters the task of algorithm design. Extending the basic PDEs to account for these features and for some additional physics, such as viscoelasticity, causes the PDEs to loose their conservation-law form, forcing the algorithm designer to consider methods for non-conservative systems. Much progress has been made in the last few decades in the design of numerical method for evolutionary PDEs, notably hyperbolic equations. See for example the textbooks of Godlewski and Raviart [7], LeVeque [10] and Toro [22,23]. Such advances have permeated into various fields of application, including computational haemodynamics [4, 15]. In spite of significant progress in this field there are still plenty of challenges to overcome, including the formulation of approaches that embrace both conservative and non-conservative forms of the PDEs, and ability to extend the schemes to high-order of accuracy in both space and time. Simplicity of the algorithms is a highly desirable property, provided, of course, that accuracy and robustness are not sacrificed.

In this paper we present new computational algorithms to solve the cross-sectional averaged blood flow equations for blood flow in arterial vessels obeying an elastic closure condition, the tube law. Main features of the proposed schemes include (i) their ability to admit discontinuous geometric and biomechanical parameters, (ii) their ability to treat both the conservative and non-conservative forms of the equations and (iii) simplicity. The schemes presented in this paper build upon two existing approaches. The first concerns a flux vector splitting (FVS) method, along the lines of the classical FVS schemes due to Steger and Warming [19], van Leer [32], Liou and Steffen [11], Zha and Bilgen [33], and the more recent FVS of Toro and Vázquez [30]. An analogous, but different, splitting scheme was proposed in [21] to discretise the equations of compressible multiphase flow; in this approach the flux vector was split into several subsets of components, one for each phase, in order to compute a single numerical flux for all phases and then update all phases simultaneously. One may term this splitting approach phase splitting. As the term splitting appears in several contexts, a word of caution is in order. First, these flux vector splitting approaches are distinct from time-splitting methods to treat source terms, or dimensional