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A Nonconforming Virtual Element Method for the Elliptic Interface Problem

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Abstract. In this paper, we propose a nonconforming virtual element method for the elliptic interface problem based on an unfitted polygonal mesh. On interface elements, the intersecting points of the interface and the edges of elements are considered as additional nodes of the mesh, and then we present a virtual element space satisfying the interface conditions. On non-interface elements, we use the usual nonconforming virtual element. By employing a computable operator, we introduce a discrete scheme and obtain optimal convergence results which are independent of the contrast of the coefficients. Numerical examples are presented to validate the theoretical results.

AMS subject classifications: 65N15, 65N30

Key words: Nonconforming, virtual element, elliptic interface problem, unfitted mesh.

1. Introduction

It is known that physical systems such as blood flow models, heat conduction problems, and groundwater pollution are coupled through interfaces. However, the discontinuity of the system coefficients on the interface, may influence the performance of the numerical methods employed in the problem solution.

We note that finite element methods for interface problems can be classified as fitted- and unfitted-mesh methods. Based on fitted-mesh, the standard finite element method can obtain the optimal convergence [7]. However, it is costly to generate a good quality mesh, especially if the interface is complex or the interface moves. So the unfitted-mesh method has attracted a lot of attentions. The unfitted-mesh method for solving interface

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problems is mainly divided into two categories. One is to double the degrees of freedom on interface elements and weakly impose the interface conditions in a discrete bilinear form [4,9,10,22,24,27,30,40]. The other one is to modify the finite element basis functions on the interface elements to approximately satisfy interface conditions [19, 23, 29, 31–34].

The virtual element method (VEM), proposed by L. Beirão da Veiga *et al.* [5], is an effective numerical method for partial differential equation problems. It has been successfully applied to various problems, such as elastic problems [6, 41], control problems with Darcy constraint [39], Stokes problems [2, 11, 37], Helmholtz problems [35], elliptic hemivariational inequalities [20], etc. Actually, VEM can be regarded as a generalization of the finite element method. The main characteristics of this method are that there are no explicit expressions of the basis functions usually, which makes it easier to extend to higher order approximation, and it can naturally deal with hang nodes, making it suitable for geometrically complex problems. In fact, numerical experiments have shown that the VEM is robust to mesh distortion. Brenner and Sung [8] established the optimal error estimates for VEMs discretization of a model Poisson problem on polygonal or polyhedral meshes with small edges or faces.

There are a few papers discussing VEM for solving interface problems. In [13, 14], a virtual element method is proposed to solve the Maxwell interface problem in two dimensions and electromagnetic interface problem in three dimensions respectively. Immersed virtual element methods for elliptic interface problems in two dimensions is analyzed in [15]. In [17], the authors propose an interface-fitted shape regular polytopal mesh generator and virtual element methods for elliptic interface problems. Tushar *et al.* [36] extend the analysis for the finite element method in [18] to VEM for the two dimensional elliptic interface problem and obtain nearly optimal error estimates under realistic assumptions. All the existing VEMs for the interface problem are conforming. To the best of our knowledge, there is no nonconforming VEM for interface problem in the literature.

In this paper, we introduce a nonconforming virtual element method, which is a generalization of the conforming version in [38], for the elliptic interface problem. On each non-interface element, we use the usual nonconforming virtual element in [3]. On each interface element, we regard the intersection points between each interface element and the interface as nodes of a polygon, and hereafter introduce a lowest order virtual element space satisfying the interface conditions. Then, we define a suitable discrete bilinear form, which consists of two parts, one is the projection term, behaving as a polynomial to ensure approximation, and the other one is the remainder, which is called the stabilization, see [12]. Unfortunately, on interface elements, our local virtual element spaces do not contain linear polynomials. We introduce a piecewise linear polynomial space that weakly satisfies the interface conditions and a computable projection-like operator to the piecewise linear polynomial space. We prove that the error with respect to the energy norm for the proposed method is optimal, and does not depend on the high contrast of the coefficients.

An outline of the paper is as follows. In Section 2, we introduce the model problem and some notations. The virtual element space and the discrete bilinear form are presented in Section 3. In Section 4, the error equation and error estimates are presented. Finally, Section 5 contains numerical examples aimed to verify the theoretical results.