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On Pricing Options Under Two Stochastic Volatility Processes

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Abstract. From the Black-Scholes option pricing model, this work evaluates the evolution of the mathematical modelling into the double stochastic volatility model that studies the optimization performance in partial differential equation (PDE) methods. This paper focuses on the calibration and numerical methodology processes to derive the comparison of the Heston and the double Heston models to design a more efficient numerical iterative splitting method. Through Li and Huang's iterative splitting method, the numerical results conclude that the mixed method reduces the overall computational cost and improves the convergence of the iterative process while maintaining the simplicity, flexibility and interpretability of PDE methods.

AMS subject classifications: 65M06, 90C26, 35C20, 35K25

Key words: Iterative splitting, asymptotic expansion, calibration, stochastic volatility.

1. Introduction

Based on the study of the widely known Black-Scholes option pricing model [2], the assumption due to the pricing of the asset follows a geometric Brownian motion with constant drift and volatility that leads to the understanding of its many limitations. It was noted that the Black-Scholes option pricing model results in the deficiency that the Normal distribution is inadequate to pick up the skewness and kurtosis observed in real financial data, and the observed market price for out-of-the-money put and in-the-money call options are higher than the Black-Scholes price. This discrepancy is known as the volatility skew. To overcome this problem, the idea that introducing a stochastic volatility process to drive the asset is developed, which allows for a negative correlation between the level of asset return and its variance to capture the leverage effect. In this regard, we refer to Wiggins [37], Hull and White [23], Scott [34], Chesney and Scott [6], Schobel and Zhu [33] and Heston [21].

Starting from the single stochastic volatility model, different extension forms have been proposed. For example, adding stochastic interest [18] to describe dynamic interest rates,

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and jump diffusions [24, 32] to reflect unexpected events. Recent modification is to incorporate regime switching mechanics into the stochastic volatility model [19, 20, 28] to capture nonlinear mean-reverting.

The single stochastic volatility models can generate the volatility smile, but they are limited in modelling the relationship between the slope of the smile and the volatility level. The restriction in describing volatility surface comes from the constant correlation between variance and asset returns. To modify the above issue, one such paper is the most recent work by Christoffersen *et al.* [8] who showed empirically with the aid of principal component analysis that this model offers flexibility in generating the smile effect and capturing the volatility term structure. This new model has two stochastic variances, the first one has a high mean reversion and the second one has a lower mean reversion, describing the correlation between short-term returns and variance and the correlation between long-term returns and variance, respectively. It has also been shown empirically by Fonseca *et al.* [10] that multiple-factor stochastic volatility models offer more consistent option prices as compared to single-factor models. Li and Zhang [25] verified that two stochastic volatilities add more flexibility to model the volatility structure and make it a better empirical fit to European option prices by analyzing an index option dataset.

In this study, the assumption that the underlying asset is driven by two stochastic variance processes of Christoffersen *et al.* [8] type is expressed in the form of stochastic differential equations (SDEs). Under the risk-neutral probability measure, the option price can be expressed as the conditional expectation of the present value of the payoff specified by the contract which is also a solution to PDE based on the Feynman-Kac theorem. The PDE corresponding to the SDEs for the option price can be derived by using the Itô lemma and constructing a suitable portfolio of hedging assets [14].

Several methods have been proposed to solve the SDE problem. Abbas-Turki and Lapeyre [1] proposed a Monte Carlo based method, and Canhanga *et al.* [3] introduced an efficient Monte Carlo simulation method based on the double stochastic volatility model. Although such a method is flexible and easy to implement, it is very slow and loads of simulations are needed to get an accurate answer. Another numerical approach is to obtain the distribution that the underlying asset price follows through the model assumptions, and after calculating its corresponding eigenfunction, the Fast Fourier Transform (FFT) can be used to find the price of the option. Gauthier and Possamai [15] corrected the analytical call option price formula given by Christoffersen *et al.* [8]. These methods are straightforward to use, while the process of deriving the formula is cumbersome and time-consuming. Besides, the expression of these formulas includes integral and need numerical methods to obtain final results which may vary under different discretization schemes.

Many scholars are transforming three SDEs into the PDE first and giving numerical scheme to approximate the corresponding solution. Costabile *et al.* [9] presented a binomial pyramid method based on the tree method, the continuous distribution into a discrete distribution, which is much easier to understand and implement. However, an exponential number of nodes makes the model extremely complex. Asymptotic analysis is applied in Canhanga *et al.* [4], which gave first- and second-order asymptotic expansions on European option prices. Recently, Zhang and Feng [38] calculated the American option price under