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ON THE CONVERGENCE OF β-SCHEMES

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Abstract. Yang's wavewise entropy inequality [19] is verified for β-schemes which, when $m = 2$ and under a mild technique condition, guarantees the convergence of the schemes to the entropy solutions of convex conservation laws in one-dimensional scalar case. These schemes, constructed by S. Osher and S. Chakravarthy [13], are based on unwinding principle and use E-schemes as building blocks with simple flux limiters, without which all of them are even linearly unstable. The total variation diminishing property of these methods was established in the original work of S. Osher and S. Chakravarthy.

Key words. Conservation laws, fully-discrete β-schemes, entropy convergence.

1. Introduction

We consider numerical approximations to the scalar conservation laws

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where $f \in C^1(\mathbb{R})$ is convex, and $u_0 \in BV(\mathbb{R})$. Here BV stands for the subspace of L^1_{loc} consisting of functions with bounded total variation. For the numerical methods concerned, let $\lambda = \frac{\tau}{h}$, where h and τ are spatial and temporal steps respectively, and $u_k^n = u(x_k, t_n)$ be the nodal values of the piecewise constant mesh function $u_h(x,t)$ approximating the solution of (1). The numerical schemes admit conservative form

$$u_k^{n+1} = H(u_{k-p}^n, \cdots, u_{k+p}^n; \lambda) = u_k^n - \lambda(g_{k+\frac{1}{2}}^n - g_{k-\frac{1}{2}}^n), \tag{2}$$

where the numerical flux g is given by

$$g_{k+\frac{1}{2}}^n = g_{k+\frac{1}{2}}[u_k^n], \tag{3}$$

and

$$g_{k+\frac{1}{2}}[v] = g(v_{k-p+1}, v_{k-p+2}, \cdots, v_k, \cdots, v_{k+p}), \tag{4}$$

for any data $\{v_j\}$. The function g is Lipschitz continuous with respect to its $2p$ arguments and is *consistent* with the conservation law in the sense that

$$g(u, u, \cdots, u) \equiv f(u). \tag{5}$$

The schemes that we are interested in are special cases of the general β-schemes when $m = 2$, which were introduced by S. Osher and S. Chakravarthy [1, 13] in the 80s. The entire families of β-schemes are defined for $0 < \beta \leq (m\binom{2m}{m})^{-1}$, where m is an integer between 2 and 8. These schemes are $2m-1$ order accurate (except at isolated critical points), variation diminishing, $2m+1$ point band width, conservative approximations to scalar conservation laws. Although the numerical results have been shown [1, 13] to be extremely good, the entropy convergence of these schemes have been open. Our goal of this paper is to show that, when $m = 2$,

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β-schemes indeed persist entropy consistency for homogeneous scalar convex conservation laws. The proof of the convergence is an application of Yang's wavewise entropy inequality (WEI) framework [19], of which he has used to establish the entropy convergence of generalized MUSCL schemes and a class of schemes using flux limiters discussed by Sweby [15]. Recently, by using Yang's convergence criteria that derived from his WEI framework, the author [9, 6] has shown the entropy convergence of van Leer's flux limiter schemes, as well as Osher-Chakravarthy's α schemes for $m = 2$ [1, 13]. The corresponding convergence results of Yang and the author, for semi-discrete schemes, can be found in [7, 8, 10, 18, 17].

The paper is organized as follows. In section 2, we review the notions of the extremum paths, and then we establish the extremum traceableness of general TVD schemes, which is necessary for analyzing the entropy convergence of the schemes that will be given in the next section. In section 3, we present one of Yang's convergence criteria with weaker condition, an important entropy estimate, and finally the main result.

Now we introduce the β-schemes for the case of $m = 2$. Throughout the paper, to improve the readability, we use the shorthand notations of $f_k^n := f(u_k^n)$, $\Delta u_{k\pm\frac{1}{2}}^n = \pm(u_{k\pm1}^n - u_k^n)$, and $f_{k\pm\frac{1}{2}}^n := \Delta f_{k\pm\frac{1}{2}}^n = \pm(f_{k\pm1}^n - f_k^n)$. Also, whenever there is no ambiguity in the context, we employ the simplified notations: $u^k := u_k^{n+1}$, $u_k := u_k^n$, $f_k := f_k^n$, and $f_{k\pm\frac{1}{2}}^{\pm} := (f_{k\pm\frac{1}{2}}^n)^{\pm}$, where k and n are the spatial and temporal indexes respectively.

Let $g_{k+\frac{1}{2}}^E := g^E(u_k^n, u_{k+1}^n)$ be the flux of an E-scheme [14] that is characterized by

$$\operatorname{sgn}(\, u_{k+1}^n - u_k^n\,)[g_{k+\frac{1}{2}}^E - f(u)\,] \leq 0, \tag{6}$$

for all u in between u_k^n and u_{k+1}^n. Then the flux differences are defined by

$$f_{k+\frac{1}{2}}^+ = f_{k+1} - g_{k+\frac{1}{2}}^E, \tag{7}$$

and

$$f_{k+\frac{1}{2}}^- = g_{k+\frac{1}{2}}^E - f_k. \tag{8}$$

At the time level $t = t^n$, for all k, we define a series of local CFL numbers

$$\nu_{k+\frac{1}{2}}^+ = \frac{\lambda f_{k+\frac{1}{2}}^+}{u_{k+\frac{1}{2}}}, \qquad \nu_{k+\frac{1}{2}}^- = \frac{\lambda f_{k+\frac{1}{2}}^-}{u_{k+\frac{1}{2}}}. \tag{9}$$

Clearly, we have $\nu_{k+\frac{1}{2}}^+ \geq 0$ and $\nu_{k+\frac{1}{2}}^- \leq 0$. For convenience, we also set the ratios

$$r_k^+ = \frac{f_{k-\frac{1}{2}}^+}{f_{k+\frac{1}{2}}^+}, \qquad r_k^- = \frac{f_{k+\frac{1}{2}}^-}{f_{k-\frac{1}{2}}^-}. \tag{10}$$

The "minmod" operator is given by

$$\operatorname{minmod}(x, y) = \begin{cases} x, \text{ if } |x| \leq |y| \text{ and } xy > 0, \\ y, \text{ if } |x| > |y| \text{ and } xy > 0, \\ 0, \text{ if } xy \leq 0, \end{cases} \tag{11}$$