Analysis in Theory and Applications
Anal. Theory Appl., Vol. **29**, No. 4 (2013), pp. 390-400

DOI: 10.4208/ata.2013.v29.n4.8

New Inequalities on L_γ-Spaces

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Received 20 September 2013; Accepted (in revised version) 19 November 2013

Available online 31 December 2013

Abstract. We consider for a fixed μ, the class of polynomials

$$P_{n,\mu,s}:=\Big\{P(z)=z^s\Big(a_nz^{n-s}+\sum_{j=\mu}^{n-s}a_{n-j}z^{n-j-s}\Big);\ 1\le\mu\le n-s\Big\}$$

of degree n, having all zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin. In this paper, we have obtained inequalities in the reverse direction for the above class of polynomials. Besides, extensions of some Turan-type inequalities for the polar derivative of polynomials have been considered.

Key Words: Polynomial, Zygmund inequality, polar derivative.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction

Let P_n be the class of polynomials $P(z)$ of degree atmost n. For $P\in P_n$, define

$$\|P\|_\gamma:=\Big\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^\gamma\Big\}^{\frac{1}{\gamma}},\quad 1\le\gamma<\infty,$$
$$\|P\|_\infty:=Max_{|z|=1}|P(z)|\quad\text{and}\quad m:=Min_{|z|=k}|P(z)|.$$

A famous result of S. Bernstein [14] states that if $P\in P_n$, then

$$\|P'\|_\infty\le n\|P\|_\infty. \tag{1.1}$$

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The result is sharp and equality holds in (1.1) for the polynomial $P(z)=\alpha z^n$, where $\alpha\neq 0$. Inequality (1.1) was extended to L_γ-norm by Zygmund [16] who proved that

$$\|P'\|_\gamma \leq n\|P\|_\gamma, \quad \gamma\geq 1. \tag{1.2}$$

For the class of polynomials $P\in P_n$ and $P(z)$ having no zero in $|z|<k$, $k\geq 1$, Malik [12] proved

$$\|P'\|_\infty \leq \frac{n}{1+k}\|P\|_\infty. \tag{1.3}$$

The result is best possible and equality holds in (1.3) for the polynomial $P(z)=(z+k)^n$.

As a generalization of (1.3) Bidkham and Dewan [6] proved that, if $P\in P_n$ and $P(z)$ having no zero in $|z|<k$, $k\geq 1$, then for $1\leq R\leq k$,

$$Max_{|z|=R}|P'(z)|\leq \frac{n(R+k)^{n-1}}{(1+k)^n}\|P\|_\infty. \tag{1.4}$$

The result is best possible and equality holds in (1.4) for the polynomial $P(z)=(z+k)^n$.

On the other hand, it was shown by Turan [15] that if $P(z)$ has all its in $|z|\leq 1$, then

$$\|P'\|_\infty \geq \frac{n}{2}\|P\|_\infty. \tag{1.5}$$

The result is best possible and equality holds in (1.5) for every such polynomial having all its zeros on $|z|=1$.

Inequality (1.5) was refined by Aziz and Dawood [2] by proving under the same hypothesis that

$$\|P'\|_\infty \geq \frac{n}{2}\Big\{\|P\|_\infty + m\Big\}. \tag{1.6}$$

Again the result is best possible and equality holds in (1.6) for $P(z)=\alpha z^n+\beta$, where $|\alpha|=|\beta|$.

As an extension of (1.5), Malik [12] showed that if $P(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, then

$$\|P'\|_\infty \geq \frac{n}{1+k}\|P\|_\infty, \tag{1.7}$$

whereas if $P(z)$ has all its zeros in $|z|\leq k, k\leq 1$ with s-fold zero at the origin, then Aziz and Shah [5] proved that

$$\|P'\|_\infty \geq \frac{n+sk}{1+k}\|P\|_\infty. \tag{1.8}$$

Both the estimates (1.7) and (1.8) are also sharp and equality in (1.7) holds for the polynomial $P(z)=(z+k)^n$ whereas equality in (1.8) holds for the polynomial $P(z)=z^s(z+k)^{n-s}$, $0\leq s\leq n$.