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An Effective Indirect Trefftz Method for Solving Poisson Equation in 2D

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Abstract. In the solution domain, the inhomogeneous part of Poisson equation is approximated with the 5-order polynomial using Galerkin method, and the particular solution of the polynomial can be determined easily. Then, the solution of the Poisson equation is approximated by superposition of the particular solution and the T-complete functions related to the Laplace equation. Unknown parameters are determined by Galerkin method, so that the approximate solution is to satisfy the boundary conditions. Comparison with analogous results of others numerical method, the two calculating examples of the paper indicate that the accuracy of the method is very high, which also has a very fast convergence rate.

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1 Introduction

The Poisson equation has universal applicability, ranging from gravity, material physics and engineering to biochemical studies [1-4]. The exact analytical solutions, although useful in analyzing the phenomena under consideration, are almost impossible to achieve once the field equations, the boundary conditions or the geometry of the boundary deviate from being simple. Therefore, studying the numerical method is very meaningful.

The three main numerical techniques are the Finite Difference Method (FDM), the Finite Element Method (FEM), and the Boundary Element Method (BEM). The Finite Difference Method, which has been extensively applied to problems of fluid dynamics as well as stress analysis solves a problem as a pointwise approximation of its governing

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equations. However, the method is difficult to apply to problems involving irregular or complex geometric, and the solving process is always too complicate. The Finite Element Method can be easily applied for solving problems with complex geometries and has been successfully used for solving problems in fluid mechanics, potential flow, rock and solid mechanics and acoustics. However, since the method only provides an approximation to the solution, it is often necessary to use a large number of very small elements to obtain accurate results. By employing a very fine mesh, the number of nodal points and hence the size of the stiffness matrix, can increase rapidly, occupying a large amount of the computer's available storage space. The size of the global stiffness matrix, and hence the problems associated with it, can be reduced by employing a Boundary Element Method which reduces the dimensionality of the problem by one. Drawbacks of this method arise from the non-uniqueness of locations of the chosen singularities. In the numerical treatment, the method may be highly ill-conditioned.

Trefftz method is the boundary-type solution procedures using T-complete functions satisfying the governing equation. It was firstly proposed by Trefftz in 1926 ([5]). When applying the Trefftz method to the analysis of the boundary value problem of the Poisson equation, there is great difficulty due to the inhomogeneous term of the governing equation. If the inhomogeneous term is simple function, e.g. the unknown function is not included in it, the particular solution for the term can be derived easily and the T-complete functions can be defined from the homogeneous and particular solutions of the problem. If not so, e.g. the inhomogeneous term includes the unknown function, the particular solutions cannot be derived directly. In order to overcome this problem, this paper employs the Galerkin analysis scheme for the analysis. First, an inhomogeneous term is approximated by the polynomial function in the Cartesian-coordinates using Trefftz-type weighted residual method to determine the particular solution related to the function. Then, the solution of the Poisson equation is approximated by superposition of the particular solution and the T-complete functions related to the Laplace equation. Unknown parameters are determined using Galerkin method by letting the residual to be zero so that the approximate solution is to satisfy the boundary conditions. This formulization is based on the so-called indirect method. Since, in this case, no discretization is needed as well as the boundary integrals are regular, the algorithm is simple. Finally, the present method is applied to two numerical examples in order to examine the numerical properties of the present method.

2 Theoretical foundations of the method

Considering a two-dimensional Poisson equation

$$\begin{cases} \nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = b(x,y,u), \qquad (x,y)\in\Omega, \\ u=\bar{u}, \qquad (x,y)\in\Gamma_1, \\ \dfrac{\partial u}{\partial n} = \bar{q}, \qquad (x,y)\in\Gamma_2, \end{cases} \tag{2.1}$$