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On the Marchenko System and the Long-time Behavior of Multi-soliton Solutions of the One-dimensional Gross-Pitaevskii Equation

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Abstract. We establish a rigorous well-posedness results for the Marchenko system associated to the scattering theory of the one dimensional Gross-Pitaevskii equation (GP). Under some assumptions on the scattering data, these well-posedness results provide regular solutions for (GP). We also construct particular solutions, called N-soliton solutions as an approximate superposition of traveling waves. A study for the asymptotic behaviors of such solutions when $t\to\pm\infty$ is also made.

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1 Introduction

The nonlinear Schrödinger equation (NLS) is given by

$$i\partial_t u+\Delta u+f(|u|^2)u=0,\quad x\in\mathbb{R}^N,\quad t\in\mathbb{R}, \tag{1.1}$$

where $f:\mathbb{R}^+\to\mathbb{R}$ is some smooth function and $N\in\mathbb{N}$. We often complete this equation by the boundary condition at infinity

$$\lim_{|x|\to+\infty}|u(t,x)|^2=\rho_0,$$

where the constant $\rho_0\geq 0$ is such that $f(\rho_0)=0$. Equation (NLS) is called focusing when $f'(\rho_0)>0$ and defocusing when $f'(\rho_0)<0$.

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These equations are widely used as models in many domains of mathematical physics, especially in non-linear optic, superfluidity, superconductivity and condensation of Bose-Einstein (see for example [1–3]). In non-linear optic case, the quantity $|u|^2$ corresponds to a light intensity, in that of the Bose-Einstein condensation, it corresponds to a density of condensed particles.

In what follows, we extend by Gross-Pitaevskii equation the one-dimensional non-linear Schrödinger equation with cubic nonlinearity, namely

$$i\partial_t u+\partial_x^2 u+(1-|u|^2)u=0, \tag{GP}$$

and for which $\rho_0=1$.

Multi-soliton solutions

In the early seventies, Shabat and Zakharov discovered an integrable system for the one-dimensional cubic Schrödinger equation. They also proposed a semi-explicit method of resolution through the theory of inverse scattering. Their results were presented in two short articles, the first one is devoted to the focusing case [4] and the second one is devoted to the defocusing case (i.e. to the Gross-Pitaevskii equation (GP)) [5]. A similar structure was discovered a few years earlier by, Green, Kruskal and Miura [6] for Korteweg - de Vries equation

$$\partial_t v+\partial_x^3 v-6v\partial_x v=0. \tag{KdV}$$

These equations have in common special solutions called *solitons*, which play an important role in dynamics in the long time. In the following, we call **soliton** a progressive wave solution, i.e. a solution of the form

$$u(t,x)=U(x-ct),$$

where U is the wave profile and c its speed. For the Gross-Pitaevskii equation (GP), such solutions (of finite Ginzburg-Landau energy except for trivial constant solutions) exist if and only if the speed c satisfies $c\in(-\sqrt{2},\sqrt{2})$. In this case, for a fixed speed, the profile is unique modulo the invariants of the equation: specifically

$$u(t,x)=e^{i\theta_0}U_c(x-x_0-ct),$$

where U_c is explicitly given by the formula

$$U_c(x)=\sqrt{1-\frac{c^2}{2}}\tanh\left(\sqrt{1-\frac{c^2}{2}}\frac{x}{\sqrt{2}}\right)+i\frac{c}{\sqrt{2}}, \tag{1.2}$$

and θ_0, $x_0\in\mathbb{R}$ are arbitrary and reflect the invariance by renormalization of the phase and translation.

We will prove the following result: