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# Existence and Nonexistence of Weak Positive Solution for a Class of $p$-Laplacian Systems

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**Abstract.** In this work, we are interested to obtain some result of existence and nonexistence of positive weak solution for the following $p$-Laplacian system

$$\begin{cases} -\Delta_{p_i} u_i = \lambda_i f_i(u_1,\cdots,u_m), & \text{in } \Omega, \ i=1,\cdots,m, \\ u_i = 0, & \text{on } \partial\Omega, \ \forall i=1,\cdots,m, \end{cases}$$

where $\Delta_{p_i} z = \operatorname{div}\left(|\nabla z|^{p_i-2}\nabla z\right)$, $p_i \geq 1, \lambda_i, 1 \leq i \leq m$ are a positive parameter, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The proof of the main results is based to the method of sub-supersolutions.

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**Key Words**: Positive solutions; sub-supersolutions; elliptic systems.

## 1 Introduction

In this paper, we are concerned with the existence and nonexistence of positive weak solution to the quasilinear elliptic system

$$\begin{cases} -\Delta_{p_i} u_i = \lambda_i f_i(u_1,\cdots,u_m), & \text{in } \ \Omega, \ 1 \leq i \leq m, \\ u_i = 0, & \text{on } \ \partial\Omega, \ \forall i, \ 1 \leq i \leq m, \end{cases} \tag{1.1}$$

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where $\Delta_p z = \mathrm{div}\big(|\nabla z|^{p-2}\nabla z\big)$, $p \geq 1, \lambda_i, 1 \leq i \leq m$ are a positive parameter, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We prove the existence of a positive weak solution for $\lambda_i > \lambda_i^*, 1 \leq i \leq m$ when

$$\lim_{t\to+\infty}\frac{f_i(\hat{t})}{t^{p_i-1}}=0, \quad \hat{t}=\underbrace{(t,\cdots,t)}_{m\text{ time}}, \ \forall i,\ 1\leq i\leq m.$$

Problems involving the $p$-Laplacian arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mapping as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids.

Hai, Shivaji [1] studied the existence of positive solution for the $p$-Laplacian system

$$\begin{cases} -\Delta_p u=\lambda f(v), & \text{in } \Omega, \\ -\Delta_p v=\lambda g(u), & \text{in } \Omega, \\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $f(s), g(s)$ are the increasing functions in $[0,\infty)$ and satisfy

$$\lim_{s\to+\infty}\frac{f\Big(M(g(s))^{\frac{1}{p-1}}\Big)}{s^{p-1}}=0, \qquad M>0,$$

the authors showed that the problem (1.2) has at least one positive solution provided that $\lambda > 0$ is large enough.

In [2], the author studied the existence and nonexistence of positive weak solution to the following quasilinear elliptic system

$$\begin{cases} -\Delta_p u=\lambda u^{\alpha}v^{\gamma}, & \text{in } \Omega, \\ -\Delta_q u=\lambda u^{\delta}v^{\beta}, & \text{in } \Omega, \\ u=v=0 \text{ on }, & \partial\Omega. \end{cases} \tag{1.3}$$

The first eigenfunction is used to construct the subsolution of problem (1.3), the main results are as follows:

(i) If $\alpha,\beta \geq 0, \gamma,\delta > 0, \theta=(p-1-\alpha)(q-1-\beta)-\gamma\delta > 0$, then problem (1.3) has a positive weak solution for each $\lambda > 0$;

(ii) If $\theta = 0$ and $p\gamma = q(p-1-\alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (1.3) has no nontrivial nonnegative weak solution.

## 2 Definitions and notations

Throughout this paper, we let $X$ be the Cartesian product of $m$ spaces $W_0^{1,p_i}(\Omega)$ for $1 \leq i \leq m$, i.e., $X = W_0^{1,p_1}(\Omega)\times\cdots\times W_0^{1,p_m}(\Omega)$. We give the definition of weak solution and sub-super solution of (1.1) as follows.