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Existence of Nontrivial Weak Solutions to Quasi-Linear Elliptic Equations with Exponential Growth

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Abstract. In this paper, we study the existence of nontrivial weak solutions to the following quasi-linear elliptic equations

$$-\triangle_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta},\quad x\in\mathbb{R}^n\ \ (n\geqslant 2),$$

where $-\triangle_n u=-\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, $0\leqslant\beta<n$, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $f(x,u)$ is continuous in $\mathbb{R}^n\times\mathbb{R}$ and behaves like $e^{\alpha u^{\frac{n}{n-1}}}$ as u $\to+\infty$.

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1 Introduction

Consider nonlinear elliptic equations of the form

$$-\triangle_p u=f(x,u),\qquad \text{in }\Omega, \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, and $-\triangle_p u=-$ div $(|\nabla u|^{p-2}\nabla u)$. Brézis [1], Brézis-Nirenberg [2] and Bartsh-Willem [3] studied this problem under the assumptions that $p=2$ and $|f(x,u)|\leqslant c(|u|+|u|^{q-1})$. Garcia-Alonso [4] studied this problem under the assumptions that $p\leqslant n$ and $p^2\leqslant n$. When $\Omega=\mathbb{R}^n$ and $p=2$, Kryszewski-Szulkin [5], Alama-Li [6], Ding-Ni [7] and Jeanjean [8] studied the following equations in stead of (1.1):

$$-\triangle u+V(x)u=f(x,u),\quad \text{in }\mathbb{R}^n.$$

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In this paper we consider quasi-linear elliptic equations in the whole Euclidean space

$$-\triangle_n u+V(x)|u|^{n-2}u=\frac{f(x,u)}{|x|^\beta},\qquad x\in\mathbb{R}^n\ \ (n\geqslant 2), \tag{1.2}$$

where $-\triangle_n u=-\mathrm{div}(|\nabla u|^{n-2}\nabla u)$, $0\leqslant\beta<n$, $V:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, $f(x,u)$ is continuous in $\mathbb{R}^n\times\mathbb{R}$ and behaves like $e^{\alpha u^{\frac{n}{n-1}}}$ as $u\to+\infty$.

D. Cao [9] and Cao-Zhang [10] studied problem (1.2) in the case $n=2$ and $\beta=0$. Panda [11], do Ó et al. [12,13] and Alevs-Figueiredo [14] studied problem (1.2) in general dimension and $\beta=0$. When $\beta\neq 0$, (1.2) was studied by Adimurthi-Yang [15], do Ó et al. [16], Yang [17], Zhao [18], and others. Similar problems in $\mathbb{R}^4$ or complete noncompact Riemannian manifolds were also studied by Yang [19,20].

We define a function space

$$E\triangleq\left\{u\in W^{1,n}(\mathbb{R}^n):\int_{\mathbb{R}^n}V(x)|u|^n\mathrm{d}x<\infty\right\}$$

with the norm

$$\|u\|\triangleq\left\{\int_{\mathbb{R}^n}(|\nabla u|^n+V(x)|u|^n)\mathrm{d}x\right\}^{\frac{1}{n}}. \tag{1.3}$$

We say that $u\in E$ is a weak solution of problem (1.2) if for all $\varphi\in E$ we have

$$\int_{\mathbb{R}^n}(|\nabla u|^{n-2}\nabla u\nabla\varphi+V(x)|u|^{n-2}u\varphi)\mathrm{d}x=\int_{\mathbb{R}^n}\frac{f(x,u)}{|x|^\beta}\varphi\,\mathrm{d}x.$$

If a weak solution u satisfies $u(x)\geqslant 0$ for almost every $x\in\mathbb{R}^n$, we say u is positive.

Throughout this paper we assume the following two conditions on the potential $V(x)$:

(V_1) $V(x)\geqslant V_0>0$;

(V_2) The function $\frac{1}{V(x)}$ belongs to $L^{1/(n-1)}(\mathbb{R}^n)$.

We also assume that the nonlinearity $f(x,s)$ satisfies the following:

(H_1) There exist constants $\alpha_0,b_1,b_2>0$ such that for all $(x,s)\in\mathbb{R}^n\times\mathbb{R}^+$,

$$|f(x,s)|\leqslant b_1 s^{n-1}+b_2\left\{e^{\alpha_0|s|^{\frac{n}{n-1}}}-\sum_{k=0}^{n-2}\frac{\alpha_0^k|s|^{\frac{kn}{n-1}}}{k!}\right\};$$

(H_2) There exists $\mu>n$, such that for all $x\in\mathbb{R}^n$ and $s>0$,

$$0<\mu F(x,s)\equiv\mu\int_0^s f(x,t)\mathrm{d}t\leqslant sf(x,s);$$

(H_3) There exist constants $R_0,M_0>0$, such that for all $x\in\mathbb{R}^n$ and $s>R_0$,

$$F(x,s)\leqslant M_0 f(x,s);$$