JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **25**, No. 2, pp. 187-198

doi: 10.4208/jpde.v25.n2.5
May 2012

Existence of Positive Solutions for Kirchhoff Type Problems with Critical Exponent

SUN Yijing and LIU Xing*

School of Mathematics, Graduate University of the Chinese Academy of Sciences, Beijing 100049, China.

Received 7 October 2011; Accepted 13 March 2012

Abstract. In this paper, we consider the following Kirchhoff type problem with critical exponent

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u=\lambda u^q+u^5, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset R^3$ is a bounded smooth domain, $0<q<1$ and the parameters $a,b,\lambda>0$. We show that there exists a positive constant $T_4(a)$ depending only on a, such that for each $a>0$ and $0<\lambda<T_4(a)$, the above problem has at least one positive solution. The method we used here is based on the Nehari manifold, Ekeland's variational principle and the concentration compactness principle.

AMS Subject Classifications: 35J60, 35B33

Chinese Library Classifications: O175.25

Key Words: Kirchhoff type equation; Nehari manifold; Ekeland's variational principle; critical exponent.

1 Introduction and main results

This paper is devoted to the study of the existence of positive solutions of the following Kirchhoff-type problem with critical exponent

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 dx\right)\Delta u=\lambda u^q+u^5, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \qquad (P_\lambda^{a,b})$$

*Corresponding author. *Email addresses:* `yjsun@gucas.ac.cn` (Y. Sun), `liuxingcas@gmail.com` (X. Liu)

where, through this work, $\Omega\subset R^3$ is a bounded smooth domain, $0<q<1$, and the parameters $a,b,\lambda>0$.

It is well known that the Kirchhoff type problem has a mechanical and biological motivation (c.f. [1,2]) and has attracted the attention of many researchers after the work of Lions [3], where a functional analysis approach was proposed to attack it. The reader may consult [1–9] and the references therein, for more information about this problem.

In the case $a=1$ and $b=0$, Problem $(P_\lambda^{1,0})$ has been studied extensively. For example, Brezis et al. [10] has shown that Problem $(P_\lambda^{1,0})$ has at least one positive solution for $3<q<5$. When $0<q<1$, Ambrosetti et al. [11] has proved that there exists λ^* such that $(P_\lambda^{1,0})$ has at least two positive solutions for $\lambda\in(0,\lambda^*)$.

A natural interesting question is whether the results concerning the solutions of $(P_\lambda^{1,0})$ remain true for the problem $(P_\lambda^{a,b})$ with $b>0$. Stimulated by [4] and [12], in this paper we study problem $(P_\lambda^{a,b})$ and give some positive answers; to our knowledge, the results in this paper are new for the case $0<q<1$.

The main idea of our paper is as follows. Firstly, we consider the Nehari manifold

$$\Lambda_\lambda=\Big\{u\in H_0^1(\Omega)\,|\, <I'_\lambda(u),u>=0\Big\}, \tag{1.1}$$

where $I_\lambda(u)\in C^1(H_0^1(\Omega),R)$ is given by

$$I_\lambda(u)=\frac{a}{2}\int_\Omega|\nabla u|^2\mathrm{d}x+\frac{b}{4}\left(\int_\Omega|\nabla u|^2\mathrm{d}x\right)^2-\frac{\lambda}{1+q}\int_\Omega|u|^{1+q}\mathrm{d}x-\frac{1}{6}\int_\Omega|u|^6\mathrm{d}x.$$

Then we split Λ_λ into three parts:

$$\Lambda_\lambda^+=\left\{u\in\Lambda_\lambda\,|\,(1-q)a\|u\|_{H^1}^2+(3-q)b\|u\|_{H^1}^4>(5-q)\int_\Omega|u|^6\mathrm{d}x\right\}, \tag{1.2}$$

$$\Lambda_\lambda^0=\left\{u\in\Lambda_\lambda\,|\,(1-q)a\|u\|_{H^1}^2+(3-q)b\|u\|_{H^1}^4=(5-q)\int_\Omega|u|^6\mathrm{d}x\right\}, \tag{1.3}$$

$$\Lambda_\lambda^-=\left\{u\in\Lambda_\lambda\,|\,(1-q)a\|u\|_{H^1}^2+(3-q)b\|u\|_{H^1}^4<(5-q)\int_\Omega|u|^6\mathrm{d}x\right\}, \tag{1.4}$$

where we set $\|u\|_{H^1}=\big(\int_\Omega|\nabla u|^2\mathrm{d}x\big)^{\frac{1}{2}}$ for $u\in H_0^1(\Omega)$. Finally by Ekeland's variational principle, we can prove that $I_\lambda(u)$ has a critical point $u_\lambda\in\Lambda_\lambda^+$.

Before stating the main result, we give some constants. Throughout this paper, we denote by S_r the best Sobolev constant for the embedding of $H_0^1(\Omega)$ into $L^r(\Omega)$ for all $1<r\le 6$. Moreover, we define $T_1(a)$, K, $T_2(a)$, $T_3(a)$ and $T_4(a)$ by

$$T_1(a)=\frac{4}{5-q}\left(\frac{1-q}{5-q}\right)^{\frac{1-q}{4}}a^{\frac{5-q}{4}}S_{1+q}{}^{\frac{1+q}{2}}S_6{}^{\frac{3(1-q)}{4}}, \tag{1.5a}$$

$$K=\frac{2(3-q)}{5-q}, \tag{1.5b}$$