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The Equation $\Delta u+\nabla\phi\cdot\nabla u=8\pi c(1-he^u)$ on a Riemann Surface

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Abstract. Let M be a compact Riemann surface, $h(x)$ a positive smooth function on M, and $\phi(x)$ a smooth function on M which satisfies that $\int_M e^\phi \mathrm{d}V_g=1$. In this paper, we consider the functional

$$J(u)=\frac{1}{2}\int_M |\nabla u|^2 e^\phi \mathrm{d}V_g+8\pi c\int_M ue^\phi \mathrm{d}V_g-8\pi c\log\int_M he^{u+\phi}\mathrm{d}V_g.$$

We give a sufficient condition under which J achieves its minimum for $c\le\inf_{x\in M}e^{\phi(x)}$.

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1 Introduction

Suppose that M is a complete $n-$dimensional Riemannian manifold with metric g, assume that ϕ is a smooth real valued function on M, and $\mathrm{d}V_g$ is the Riemannian density on M, Wei and Wylie studied mean curvature and volume comparison results on the smooth measure space $(M^n,g,e^\phi \mathrm{d}V_g)$ (see [1]). This is also sometimes called a manifold with density. The corresponding Bakry-Emery Ricci tensor is defined as $\mathrm{Ric}_{-\phi}=\mathrm{Ric}-\mathrm{Hess}\phi$. The equation $\mathrm{Ric}_{-\phi}=\lambda g$ for some constant λ is the gradient Ricci soliton equation. It plays important role in the theory of the of Ricci flow. So this measure space is interesting. We are interested in the Dirichlet integral on this measure space:

$$D_\phi(u)=\int_M |\nabla u|^2 e^\phi \mathrm{d}V_g.$$

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The operator $\Delta+\nabla\phi\cdot\nabla$ is the naturally associated ($\phi-$)Laplacian which is self-adjoint with respect to the weighted measure $e^{\phi}\mathrm{d}V_g$ (see [1]). The critical point of D_ϕ is a ϕ-harmonic function which satisfies that

$$(\Delta+\nabla\phi\cdot\nabla)(u)=0.$$

In [1], Wei and Wylie proved some comparison theorems related to the ($\phi-$)Laplacian.

Let $(M,\mathrm{d}s^2)$ be a compact Riemann surface, suppose that $h(x)$ and $\phi(x)$ are smooth functions on M. Moreover $\max h>0$. For simplicity, we assume in this paper that $\int_M e^{\phi}\mathrm{d}V_g=1$.

On the measure space $(M^n,g,e^{\phi}\mathrm{d}V_g)$, we study the functional

$$J(u)=\frac{1}{2}\int_M|\nabla u|^2e^{\phi}\mathrm{d}V_g+8\pi c\int_M ue^{\phi}\mathrm{d}V_g-8\pi c\log\int_M he^{u+\phi}\mathrm{d}V_g,$$

the critical point of the functional satisfies the equation

$$\Delta u+\nabla\phi\cdot\nabla u=8\pi c-8\pi che^{u}, \tag{1.1}$$

where c is a constant, or we can rewrite it as

$$\mathrm{div}(e^{\phi}\nabla u)=8\pi ce^{\phi}-8\pi che^{u+\phi}.$$

If $\phi\equiv 0$ and $c=1$, Kazdan and Warner studied it thirty years ago [2]. They asked under what kind of conditions on h, the equation

$$\Delta u=8\pi-8\pi he^{u} \tag{1.2}$$

has a solution. For general c, the equation

$$\Delta u=8\pi c-8\pi che^{u}$$

is also studied by many authors, see [3,4].

One sees that (1.1) is a natural generalization of (1.2), when one studies the measure space $(M^n,g,e^{\phi}\mathrm{d}V_g)$ instead of $(M^n,g,\mathrm{d}V_g)$. In this paper, we study the existence of Eq. (1.1), which can be seen as the first step to understand the equation.

We will follow the paper [5], to minimize the functional $J(u)$.

By [5, Theorem 1.1], we can get that the functional

$$J_\varepsilon(u)=\frac{1}{2}\int_M|\nabla u|^2e^{\phi}\mathrm{d}V_g+8\pi(\rho-\varepsilon)\int_M ue^{\phi}\mathrm{d}V_g-8\pi(\rho-\varepsilon)\log\int_M he^{u+\phi}\mathrm{d}V_g,$$

achieves its minimum at some u_ε, where $\rho=\min_{x\in M}e^{\phi(x)}$. If $\|u_\varepsilon\|_{L_1^2(M)}$ is not bounded, we can show that the blow up point p is the minimum point of e^{ϕ}. Then after subtracting mean values, u_ε converges to some Green function $G(x,p)$ (sometimes we denote $G(x,p)=G_p(x)$ for simplicity) satisfying

$$\begin{cases} \mathrm{div}(e^{\phi}\nabla G_p)=8\pi\rho e^{\phi}-8\pi\rho\delta_p, \\ \displaystyle\int_M G_pe^{\phi}\mathrm{d}V_g=0. \end{cases} \tag{1.3}$$