JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **23**, No. 1, pp. 80-104

doi: 10.4208/jpde.v23.n1.5
February 2010

Infinitely Many Solutions for an Elliptic Problem with Critical Exponent in Exterior Domain

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Received 24 April 2009; Accepted 21 September 2009

Abstract. We consider the following nonlinear problem

$$\begin{cases} -\Delta u = u^{\frac{N+2}{N-2}}, \quad u>0 & \text{in } \mathbf{R}^N\backslash\Omega, \\ u(x)\to 0 & \text{as } |x|\to+\infty, \\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega\subset\mathbf{R}^N, N\geq 4$ is a smooth and bounded domain and n denotes inward normal vector of $\partial\Omega$. We prove that the above problem has infinitely many solutions whose energy can be made arbitrarily large when Ω is convex seen from inside (with some symmetries).

AMS Subject Classifications: 35B40, 35B45, 35J40

Chinese Library Classifications: O175.29

Key Words: Infinitely many solutions; critical exponent; exterior domain.

1 Introduction and main result

In this paper we consider the nonlinear Neumann elliptic problem

$$\begin{cases} -\Delta u - u^{\frac{N+2}{N-2}} = 0, \quad u>0 & \text{in } \mathbf{R}^N\backslash\Omega, \\ u(x)\to 0 & \text{as } |x|\to+\infty, \\ \dfrac{\partial u}{\partial n}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where n denotes interior unit normal vector and Ω is a smooth bounded domain in $\mathbf{R}^N, N\geq 4$.

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Semilinear elliptic equations involving critical Sobolev exponents have been considered by various authors, e.g., [1–6]. Such kind of equations arise in various branches of mathematics as well as physics (see, e.g., [2,7] and the reference therein). The most notorious example is *Yamabe's* problem: let (M,g) be a Riemannian manifold of dimension $N, N\geq 3$, and R' be a given function on M. Can one find a new metric g_1 on M such that R' is the scalar curvature of g_1 and g_1 is conformal to g (i.e., $g_1 = u^{\frac{4}{N-2}} g$ for some function $u>0$ on M)? This is equivalent to the problem of finding positive solution of the equation

$$-4\frac{N-1}{N-2}\Delta_g u = R' u^{\frac{N+2}{N-2}} - R(x)u \quad \text{on } M, \tag{1.2}$$

where Δ_g is Laplace-Beltrami operator on M in the g metric and $R(x)$ is the scalar curvature of (M,g). In case M is compact, Eq. (1.2) has been considered by many authors, see [7] for a survey on its development and a brief history. In the special case where $M=\mathbf{R}^N$ and g is the usual metric we have $R\equiv 0$ and the equation is reduced to

$$\Delta u + R' u^{\frac{N+2}{N-2}} = 0. \tag{1.3}$$

From now on we are concerned with the case $R'\equiv$constant. Without loss of generality we may assume $R'\equiv 1$. According to [8] the functions

$$U_{\lambda,a}(x) = \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{N-2}{2}}}, \quad \lambda>0, \quad a\in\mathbf{R}^N,$$

are the only solutions to the problem

$$-\Delta u = \alpha_N u^{\frac{N+2}{N-2}}, \quad u>0 \ \text{ in } \mathbf{R}^N,$$

where $\alpha_N = N(N-2)$.

On the other hand, by Divergence Theorem there is no positive solution of the following problem

$$-\Delta u = u^{\frac{N+2}{N-2}} \ \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$. Hence it has been a matter of high interest to study the problem in exterior domain, which is Eq. (1.1). In [9], Pan and Wang proved that if the mean curvature of $\partial\Omega$ seen from inside is negative somewhere, then Eq. (1.1) has a least energy solution while Ω is a ball Eq. (1.1) has no least energy solution. A natural question is: how about higher energy solutions?

The purpose of this paper is to prove that Eq. (1.1) has infinitely many higher energy solutions while Ω is convex seen from inside. More precisely, we assume that Ω is a smooth and bounded domain satisfying the following properties: