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Remark on Exponential Decay of Ground States for N-Laplacian Equations

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Abstract. We study exponential decay property of radial ground states to a class of N-Laplacian elliptic equations in the whole space $\mathbb{R}^N$. Their decay rates as $|x|\to\infty$ are obtained explicitly.

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1 Introduction

We consider the exponential decay of radial ground states of a class of N-Laplacian elliptic equations as follows:

$$\Delta_N u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \quad N \geq 2, \tag{1.1}$$

where $\Delta_N u = \text{div}(|Du|^{N-2}Du)$ is the degenerate N-Laplacian. Here by a ground state we mean a non-negative non-trivial C^1 distribution solution of (1.1) which tends to zero at infinity. The particular interest in this problem is that the order of the Laplacian is the same as the dimension of the underlying space. For the classical case of this problem, i.e., $N=2$, (1.1) reduces to

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{1.2}$$

Under suitable conditions on f it was shown in [1] that the ground state for (1.2) satisfies

$$u(x) = u(|x|) = u(r) > 0, \quad u(0) = \max_{x\in\mathbb{R}^2} u(x),$$
$$u'(0) = 0 \quad \text{and} \quad u'(r) < 0 \quad \text{for} \quad r \in (0,\infty).$$

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Moreover,

$$\lim_{r\to\infty} u(r)\sqrt{r}e^{r}=C \tag{1.3}$$

with some constant $0<C<\infty$. Such precise estimates of asymptotes as $|x|\to\infty$ have been proved to be very useful. For applications of such estimates, readers can refer to [1–5] etc. In an earlier work [6], we considered the exponential decay of ground states of the m-Laplacian equation

$$\Delta_m u+f(u)=0 \quad \text{in } \mathbb{R}^N, \quad N>m>1$$

under certain assumptions on f. It has been known that when $1<m<N$, Pohozaev-type restrictions on the nonlinear term f are needed to show the existence of ground states [7], and for $m>N$ no growth conditions are required [8]. So we can regard (1.1) as a transition or borderline case.

Throughout this paper we make the following assumptions on f:

(f_1): f: $[0,\infty)\to\mathbb{R}$ is locally Lipschtiz continuous and there exist positive constants K and α such that

$$f(t)+Kt^{N-1}=\mathcal{O}(t^{N-1+\alpha}) \quad \text{as } t\downarrow 0.$$

(f_2): There exists $\beta>0$ such that

$$F(t)=\int_0^t f(s)\mathrm{d}s<0 \quad \text{on } (0,\beta), \quad F(\beta)=0, \quad f(t)>0 \quad \text{for } t\geq\beta.$$

(f_3): For some $\gamma\geq\beta$ we have $f\in C^1[\gamma,\infty)$ and $f'(t)\geq 0$ for $t>\gamma$.

Note that an example of f satisfying all above assumptions is $f(t)=-t^{N-1}+t^p$ with $N-1<p<\infty$. Also note that under assumptions (f_1)-(f_3), the existence of a radial ground state is guaranteed and moreover if $r=|x|$ the function $u=u(r)$ satisfying $u'(r)<0$ for all $r>0$ such that $u(r)>0$ (see [9]). Our result can be stated as follows:

Theorem 1.1. *Let $u(x)=u(r)$ be a positive radial ground state for* (1.1) *and $f(t)$ satisfies assumptions (f_1)-(f_3). Then there exists a sequence of constants $\{C_i\}$ $(i=1, 2,\cdots)$ such that for any $l=1, 2,\cdots$,*

$$\Big(-\frac{u'}{u}\Big)^{N-1}=\Big(\frac{K}{N-1}\Big)^{\frac{N-1}{N}}+\frac{C_1}{r}+\frac{C_2}{r^2}+\cdots+\frac{C_l}{r^l}+\mathcal{O}\Big(\frac{1}{r^{l+1}}\Big) \quad \text{as} \quad r\to\infty,$$

where $\{C_i\}$ $(i=1, 2,\cdots)$ are determined by

$$C_1=\frac{N-1}{N}\Big(\frac{K}{N-1}\Big)^{\frac{N-2}{N}}, \quad C_2=\frac{(N-2)C_1-\frac{N}{2(N-1)}\big(\frac{K}{N-1}\big)^{\frac{2-N}{N}}C_1^2}{N\big(\frac{K}{N-1}\big)^{\frac{N-1}{N^2}}};$$