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DECAY RATES TOWARD STATIONARY WAVES OF SOLUTIONS FOR DAMPED WAVE EQUATIONS*

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Dedicated to Professor Li Tatsien on the Occasion of His 70th Birthday

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Abstract This paper is concerned with the initial-boundary value problem for damped wave equations with a nonlinear convection term in the half space $\mathbf{R}_+$

$$\begin{cases} u_{tt} - u_{xx} + u_t + f(u)_x = 0, \quad t > 0,\ x \in \mathbf{R}_+, \\ u(0,x) = u_0(x) \to u_+, \quad \text{as } x \to +\infty, \\ u_t(0,x) = u_1(x), \quad u(t,0) = u_b. \end{cases} \tag{I}$$

For the non-degenerate case $f'(u_+) < 0$, it is shown in [1] that the above initial-boundary value problem admits a unique global solution $u(t,x)$ which converges to the stationary wave $\phi(x)$ uniformly in $x \in \mathbf{R}_+$ as time tends to infinity provided that the initial perturbation and/or the strength of the stationary wave are sufficiently small. Moreover, by using the space-time weighted energy method initiated by Kawashima and Matsumura [2], the convergence rates (including the algebraic convergence rate and the exponential convergence rate) of $u(t,x)$ toward $\phi(x)$ are also obtained in [1]. We note, however, that the analysis in [1] relies heavily on the assumption that $f'(u_b) < 0$. The main purpose of this paper is devoted to discussing the case of $f'(u_b) = 0$ and we show that similar results still hold for such a case. Our analysis is based on some delicate energy estimates.

Key Words Damped wave equation; stationary wave; asymptotic stability; decay rates; space-time weighted energy method.

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1. Introduction

In this paper, we consider the initial-boundary value problem for the following damped wave equations with nonlinear convection in the half space $\mathbf{R}_+$

$$u_{tt} - u_{xx} + u_t + f(u)_x = 0, \quad t > 0, \ x \in \mathbf{R}_+ \tag{1.1}$$

with prescribed initial and boundary conditions

$$\begin{aligned} &u(0,x) = u_0(x) \to u_+, \quad \text{as } x \to +\infty, \\ &u_t(0,x) = u_1(x), \end{aligned} \tag{1.2}$$

$$u(t,0) = u_b. \tag{1.3}$$

Here u is an unknown function of $t > 0$ and $x \in \mathbf{R}_+$, $u_+ \neq u_b$ are two given constant states and the nonlinear function $f(u) \in C^2(\mathbf{R})$ is assumed to be a strictly convex smooth function of u, namely, $f''(u) > 0$ for any u under consideration. Throughout the paper, we also assume that $f(u)$ satisfies the following sub-characteristic condition on the interval $I[u_b, u_+] \triangleq [\min\{u_b, u_+\}, \max\{u_b, u_+\}]$:

$$\left|f'(u)\right| < 1 \qquad \text{for any} \quad u \in I[u_b, u_+]. \tag{1.4}$$

For such an initial-boundary value problem, to study its large time behavior, in addition to some elementary waves such as viscous shock waves and rarefaction waves which are sufficient to describe the large time behavior of solution to the corresponding Cauchy problem, a new type nonlinear wave, the so-called stationary wave, should be taken into consideration which is due to the occurrence of the boundary. The main purpose of our present manuscript is devoted to studying the nonlinear stability of such a stationary wave $\phi(x)$ and to deducing the convergence rates of the solution $u(t,x)$ of the initial-boundary value problem (1.1)-(1.3) toward $\phi(x)$.

Recalled that $\phi(x)$ is called a stationary wave of the initial-boundary value problem (1.1)-(1.3) if it satisfies the equation

$$\phi_{xx} = f(\phi)_x, \qquad x \in \mathbf{R}_+ \tag{1.5}$$

together with the boundary and the spatial asymptotic conditions

$$\phi(0) = u_b, \qquad \lim_{x \to +\infty} \phi(x) = u_+. \tag{1.6}$$

A simple analysis tells us that $f'(u_+) \leq 0$ is a necessary and sufficient condition to guarantee the existence of solutions to the problem (1.5), (1.6) and the cases $f'(u_+) < 0$