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GLOBAL NONEXISTENCE OF THE SOLUTIONS FOR A NONLINEAR WAVE EQUATION WITH THE Q-LAPLACIAN OPERATOR*

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Abstract We study the global nonexistence of the solutions of the nonlinear q-Laplacian wave equation

$$u_{tt} - \Delta_q u + (-\Delta)^\alpha u_t = |u|^{p-2}u,$$

where $0 < \alpha \leq 1$, $2 \leq q < p$. We obtain that the solution blows up in finite time if the initial energy is negative. Meanwhile, we also get the solution blows up in finite time with suitable positive initial energy under some conditions.

Key Words q-Laplacian operator; nonlinear wave equation; global nonexistence.
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1. Introduction

We study the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta_q u + (-\Delta)^\alpha u_t = |u|^{p-2}u, & x \in \Omega, t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $2 \leq q < p$, $-\Delta_q u = -\sum\limits_{i=1}^{\infty} \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{q-2}\frac{\partial u}{\partial x_i})$, and Ω is a bounded domain in $R^n, n \geq 1$, with smooth boundary $\partial\Omega$. For this problem, H. Gao and T. F. Ma [1] had obtained the global existence of the solution when $q > p$ and with small initial data when $q \leq p$.

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When $q = 2$, with the linear damping term($\alpha = 0$), H. Levine([2, 3]) had proved the solution blows up in the finite time with negative initial energy. When $q = 2$, and the damping term is given by $|u_t|^r u_t$, here $r \geq 0$, many authors had studied the existence and uniqueness of the global solution and the blowup of the solution, see [4-6]. Our objective is to study the global nonexistence for this kind of equations with $q < p$ under a weaker damping term. For negative initial energy, we use the energy method with some modifications to [7] and [8], and obtain the global nonexistence for (1.1). For positive initial energy, we use the concavity technique developed by Levine [3] to get the global nonexistence for (1.1), this method can also be found in P. Pucci and J. Serrin [9].

The damping term we consider here is different from [10]. Since for an arbitrary $0 < \alpha \leq 1$, the condition (3d) in [10] does not always hold. For the model we consider here, by [10] we know $V = L^2(\Omega)$, $W = L^p(\Omega)$ correspondingly for our case, and $W' = L^{p'}(\Omega)$, here $\frac{1}{p'} = 1 - \frac{1}{p} > \frac{1}{2}$, and

$$\begin{aligned} &Q(t,v) = (-\Delta)^{\alpha} v, \\ &\mathcal{D}(t,v) = \int_{\Omega} (Q(t,v), v)dx = \|(-\Delta)^{\alpha/2} v\|_{L^2}^2. \end{aligned}$$

By Sobolev imbedding $W^{2\alpha,p'}(\Omega) \hookrightarrow W^{\alpha,2}(\Omega)$(see [11]) with

$$\alpha \geq n(\frac{1}{p'} - \frac{1}{2}), \qquad (*)$$

we have

$$\|(-\Delta)^{\alpha} v\|_{L^{p'}} \leq C\|(-\Delta)^{\alpha/2} v\|_{L^2},$$

here C is a constant. The above inequality just is (3d) in [10] with $\delta(t)$ being constant and $m = m' = 2$. We know the condition $(*)$ does not always hold for any given $0 < \alpha \leq 1$ and for all p satisfying the condition (2.2) in the sequel, that is (3d) in [10] does not always hold for arbitrary $0 < \alpha \leq 1$. But our results hold for any $0 < \alpha \leq 1$ and all p satisfying the condition (2.2).

Here we use standard notations. We often write $u(t)$ instead $u(t,x)$ and $u'(t)$ instead $u_t(t,x)$. The norm in $L^q(\Omega)$ is denoted by $\|\cdot\|_q$ and in $W_0^{1,q}(\Omega)$ we use the norm $\|u\|_{1,q}^q = \sum_{i=1}^{n} \|u_{x_j}\|_q^q$.

For convenience, we recall some of the basic properties of the operators used here. The degenerate operator $-\Delta_q$ is unbounded, monotone and hemicontinuous from $W_0^{1,q}(\Omega)$ to $W_0^{-1,p}(\Omega)$, where $q^{-1} + p^{-1} = 1$. The power for the Laplacian operator is defined by $(-\Delta)^{\alpha} u = \sum_{j=1}^{\infty} \lambda_j^{\alpha} (u, \varphi_j)\varphi_j$, where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ....$ and $\varphi_1, \varphi_2, \varphi_3....$ are respectively the sequence of the eigenvalues and eigenfunctions of $-\Delta$ in $H_0^1(\Omega)$. Then

$$\|u\|_{D((-\Delta)^{\alpha})} = \|(-\Delta)^{\alpha} u\|_2, \quad \forall u \in D((-\Delta)^{\alpha})$$