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ATTRACTORS FOR THE BRUSSELATOR IN R^N

Han Yongqian and Guo Boling
(Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, China)
(E-mail: han_yongqian@iapcm.ac.cn)
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Abstract We consider the reaction-diffusion system, a model of a certain chemical morphogenetic process and named Brusselator. For the Cauchy problem of this system with nondecaying initial data, the existence and uniqueness of the global solution is established. Moreover, it is proved that this system possesses a global attractor $\mathcal{A}$ in the corresponding phase space.
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1. Introduction

We consider the following Cauchy problem of reaction-diffusion system

$$u_t = d\Delta u - (b+1)u + u^2 v + a, \tag{1.1}$$

$$v_t = d\Delta v + bu - u^2 v, \qquad x \in R^N, \ \ t > 0, \tag{1.2}$$

$$u(x, t=0) = u_0(x) \geq 0, \quad v(x, t=0) = v_0(x) \geq 0, \tag{1.3}$$

a model of a certain chemical morphogenetic process due to Turing [1] and named Brusselator. Here d, a, b are strictly positive constants, $N = 1, \ 2, \ 3$.

The system (1.1), (1.2) has been extensively studied, see [2–10, etc.] and therein references. In particular, the system (1.1), (1.2) was applied to study oscillatory Turing pattern [1], stationary pattern selection and competition [3]. Hollis, Martin and Pierre [4] proved the existence of a global bounded solution of (1.1), (1.2) with initial boundary conditions in the bounded domain of R^N.

To study the problem (1.1)–(1.3), the classical Sobolev spaces (such as $H^l(R^N)$ ($l \geq 1$)) does not seem to be adequate because nondecaying initial data and a number of natural structures from the physical point of view (such as spatially periodic solutions, travelling waves, static states, etc.) are out of consideration. For example, it is obvious that the static state $u_s = a$ and $v_s = b/a$ is a solution of (1.1), (1.2), but $u_s, \ v_s \notin C(0, T; H^l(R^N))$. Therefore, we discuss the system (1.1)–(1.3) in some weighted Sobolev spaces as in [11–13, etc.].

Definition 1.1 *A function $\phi \in C_{loc}(R^N)$ is a weight function with the growth rate $\alpha \geq 0$ if the conditions $\phi(x+y) \leq C_\phi e^{\alpha|x|}\phi(y)$ and $\phi(x) > 0$ are satisfied for every $x, y \in R^N$.*

Definition 1.2 *Let ϕ be a weight function with the growth rate α. We define the spaces*

$$W_\phi^{l,p}(R^N) = \Big\{u \in \mathcal{D}'(R^N):\ \|u\|^p_{\phi,l,p} \equiv \sum_{|\beta|\leq l} \int_{R^N} \phi(x)|\partial_x^\beta u(x)|^p dx < \infty,\ \ l \in \mathcal{N}\Big\},$$

$$W_{b,\phi}^{l,p}(R^N) = \Big\{u \in \mathcal{D}'(R^N):\ \|u\|^p_{b,\phi,l,p} \equiv \sup_{x_0 \in R^N} \phi(x_0)\|u,\ B^1_{x_0}\|^p_{l,p} < \infty,\ \ l \in \mathcal{N}\Big\}.$$

We will write $W_b^{l,p}$ instead of $W_{b,1}^{l,p}$.

It is obvious that the static state u_s of (1.1), (1.2) belongs to $W_b^{l,p}(R^N)$.

For semi-linear system of parabolic equations

$$u_t = a\Delta_x u + \lambda_0 u - f(u, \nabla u) + g(x), \quad x \in \Omega, \tag{1.4}$$

$$u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \tag{1.5}$$

in an unbounded domain Ω is studied in [11–14] and therein references. If the nonlinear function $f(u, \nabla u)$ satisfies the dissipation assumption

$$f(u, \nabla u) \cdot u \geq 0, \tag{1.6}$$

the global attractor of (1.4), (1.5) has been constructed in [11–13] and therein references. Moreover, the upper bounded and the lower bounded of the Kolmogorov ϵ-entropy, the spatial complexity and spatial chaos of the global attractor for equations (1.4), (1.5) have been also studied in [11–13] and therein references. Unfortunately, the nonlinearity of the equation (1.1), (1.2) doesn't satisfy the dissipation assumption (1.6).

The well posedness of some reaction-diffusion system in R^N has been extensively studied by many authors, see [15–17, etc.] and therein references.

In this paper, for the system (1.1)–(1.3), we establish the global well-posedness of the solution in $L^\infty\big([0,\infty); W_b^{2,2}(R^N)\big)$:

Theorem 1.3 *Let $u_0,\ v_0 \in W_b^{2,2}$. Then, there exists a unique solution (u, v) of system* (1.1)–(1.3) *such that $u \geq 0,\ v \geq 0$ and $u, v \in L^\infty([0,\infty); W_b^{2,2}) \cap C([0,\infty); W_b^{1,2}) \cap W^{1,\infty}([0,\infty); L_b^2) \cap L^2_{loc}([0,\infty); W_b^{3,2}) \cap W^{1,2}_{loc}([0,\infty); W_b^{1,2}) \cap C([0,\infty); W^{2,2}_{loc}) \cap C^1([0,\infty); L^2_{loc})$. Moreover, for every two solutions $\big(u_1(t), v_1(t)\big)$ and $\big(u_2(t), v_2(t)\big)$ of the equations* (1.1), (1.2)*, the following estimate holds:*

$$\begin{aligned}&\|u_1(t) - u_2(t), B^1_{x_0}\|_{0,2} + \|v_1(t) - v_2(t), B^1_{x_0}\|_{0,2} \\ &\quad \leq Ce^{K_0 t}\big(\|u_1(0) - u_2(0)\|_{\phi^0_\epsilon,0,2} + \|v_1(0) - v_2(0)\|_{\phi^0_\epsilon,0,2}\big), \quad \forall t \geq 0,\end{aligned} \tag{1.7}$$

for some positive constants C and K_0, which are independent of $\epsilon > 0$ and $x_0 \in R^N$.