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TOTAL VERSUS SINGLE POINT BLOW-UP SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION*

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Abstract In this paper, we investigate the blow-up set of solutions of a parabolic equation with localized and non-localized reactions. We completely classify blow-up solutions into total blow-up cases and single point blow-up cases.

Key Words Single point blow-up; total blow-up; non-local problem.
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1. Introduction and Main Results

In this paper we study the following parabolic equation with localized reaction:

$$\begin{cases} u_t = \Delta u + u^p + \mathrm{e}^{u(x^*,\,t)}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $p > 0$, $x^* \in \Omega$ and Ω is a ball in $\mathbb{R}^n$ with center origin and radius R. Throughout this paper, the initial data u_0 is assumed to satisfy:

$$(A) \quad \begin{cases} u_0 \in C^2(\Omega) \cap C(\overline{\Omega}), \\ u_0(x) = u_0(r) \geq 0 \quad (r =| x |), \\ u_0'(r) < 0 \quad \text{for } r \in (0, R]. \end{cases}$$

Equation (1.1) is related to some catalysis and ignition model for compressible reactive gases, see [1,2] and references therein. It is easy to show the local existence and uniqueness of classical solution of (1.1). Moreover, the solution is radially symmetric and $u(x,t) \leq u(0,t)$. If $u_0(x)$ is sufficiently large, we can prove that the solution of (1.1) blows up in finite time. From now on, we always assume that the classical solution of (1.1) blows up in finite time and denote the blow-up time by T.

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Definition 1.1 *A point $x \in \bar{\Omega}$ is called a blow-up point if there exists a sequence (x_n, t_n) such that*

$$t_n \to T, \quad x_n \to x \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as} \quad n \to \infty.$$

In this paper we are interested in studying the set of blow-up points which is denoted by S. When the solution $u(x,t)$ of (1.1) blows up everywhere, we call this phenomena "total blow up" and when the blow up point in $\bar{\Omega}$ is only one point, we call this "single point blow up".

Many of localized problems arise in applications and have been widely studied. For more references of localized problems, please see [3–6].

Our main results are the following theorems.

Theorem 1.1 *Let $u(x,t)$ be the blow-up solution of (1.1). If $x^* = 0$, then $S = \bar{\Omega}$. Moreover, $u(x,t)$ satisfies*

$$\lim_{t\to T}[\ln(T-t) + u(x,t)] = 0.$$

We would like to point out that this result improves the estimate of [5, Theorem 4.6]. And the next two theorems give the answer to the remark (1) in the end of the paper [5].

Theorem 1.2 *Let $u(x,t)$ be the blow-up solution of (1.1). If $x^* \neq 0$, $0 < p \leq 1$, then $S = \bar{\Omega}$.*

Theorem 1.3 *Let $u(x,t)$ be the blow-up solution of (1.1). If $x^* \neq 0$, $p > 1$, then*
(i) *there exists a solution which blows up on the whole domain, that is , $S = \bar{\Omega}$;*
(ii) *there exists a solution which blows up only at the origin, that is, $S = \{0\}$;*
(iii) *there are no other blow-up phenomena except $S = \bar{\Omega}$ and $S = \{0\}$.*

Before ending this section, we state some Propositions that will be used in the later.

Proposition 1.1 ([1]) *Let u be a blow-up solution of (1.1) with blow up time T. If*

$$\int_0^T e^{u(x^*,t)}\, dt = \infty,$$

then $S = \bar{\Omega}$. If

$$\int_0^T e^{u(x^*,t)}\, dt < \infty,$$

then $S = \{0\}$.

Proposition 1.2 ([7]) *Let u be a solution of (1.1), D be a domain in Ω and G be a subset of D. Then there exists a constant $C = C(D,G) > 0$ such that if*

$$\inf_{x\in D} u_0(x) \geq h$$

for some $h > 0$, then

$$\inf_{x\in G} u(x,t) \geq Ch, \quad t \in [0,T).$$