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UNIFORM ESTIMATES AND MULTISOLVABILITY OF A SEMILINEAR ELLIPTIC EQUATION*

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Abstract In this paper, the author gives the uniform estimates for the solutions of the semilinear elliptic equation $\Delta u + \lambda e^u = 0$ with zero Dirichlet boundary condition, and then derive the uniqueness of the nonminimal solution assuming some conditions satisfied.
Key Words Semilinear elliptic equation; uniform estimates; multisolvability.
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1. Introduction and Main Result

In this paper, we study the semilinear elliptic equation:

$$\begin{cases} \Delta u + \lambda e^u = 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a real constant and $\Omega \subset R^2$ is a simply connected and bounded domain with smooth boundary. By standard elliptic theory [1], we know that $u \in C^\infty(\overline{\Omega})$ and $u(x) > 0\,\forall\, x \in \Omega$.

Equation (1.1) arises in mathematical physics and also in differential geometry and has been studied by so many works (see [2] and references there in). From these works one can see the following results:

(a)Equation(1.1) **has the spectrum set** $(0, \lambda^*][3, 4]$**, and the supremum spectrum satisfies** $\frac{2\pi}{|\Omega|} \le \lambda^* \le \frac{2}{H^2}$**, here** $|\Omega|$ **means the area of** Ω **and** H **is the maximal conformal radius of** Ω $[5, 6]$**;**

(b)$\forall\, \lambda \in (0, \lambda^*)$**,** (1.1) **has a minimal solution and at least one nonminimum solution** $[3, 4]$**;**

(c)Any solution $u(x)$ **of** (1.1) **satisfies:**

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$$1-\sqrt{1-\lambda d^2(x)/2}\le 2e^{-u(x)/2}\le 1+\sqrt{1-\lambda d^2(x)/2},$$

here $d(x)=\mathrm{dist}(x,\partial\Omega)$ [6].

If $\Omega=B$(the unit disk), especially, then it is well known that all solutions of (1.1) are symmetric to the origin and can be written explicitly as:

$$u(r)=\alpha-2\,\log(1+\frac{1}{8}\lambda\,e^{\alpha}\,r^2) \tag{1.2}$$

where $r=|x|, x\in B$, $\alpha=u(0)=\max_B|u|=2\,\log\frac{2}{1\pm\sqrt{1-\lambda/2}}$, and the α-λ figure is as following:

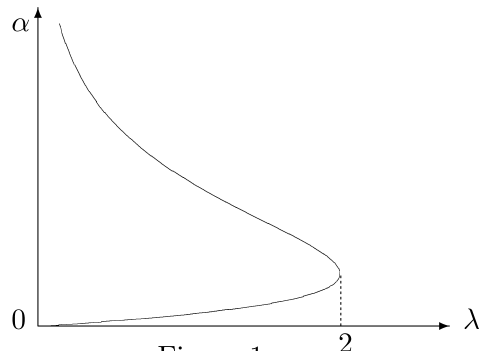

Figure 1

From Figure 1 one can see easily that now (1.1) has just one nonminimal-solution-branch.

For the general domain as above, V. H. Weston and J. L. Moseley [7, 8] have constructed a nonminimal-solution-branch by the method of singular perturbations. After that, T. Suzuki and K. Nagasaki [9] proved that, when the domain is closed to a disk in the sense of Riemann mapping, the minimal-solution-branch and the nonminimal-solution-branch are connected to each other and form also one solution-branch similar to which in Figure 1. But they also pointed out the possibility of the existence of other nonminimal-solution-branch.

In this paper, we give the uniform estimates for the solutions of (1.1) with Ω as in Definition 2.1. And then derive the uniqueness of the nonminimal solution assuming some conditions satisfied as in Theorem 3.3.

In the next section, we will give the uniform estimates of the solutions of (1.1), which has been raised as a conjecture in some articles [6, 10]. By applying this uniform estimate, the uniqueness of nonminimal solution of (1.1) is proved in the third section with the domains closed to the unit disk under some conditions of the boundary and the parameter. As a corollary, given in the forth section, we show that (1.1) has indeed only one nonminimal solution when the parameter λ approaches to 2, which means that the solution-branch can not bifurcate near the turning point.