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A HARNACK INEQUALITY APPROACH TO THE INTERIOR REGULARITY GRADIENT ESTIMATES OF GEOMETRIC EQUATIONS*

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Abstract In this paper we prove the gradient estimates for fully nonlinear geometric equation using a normal perturbation techniques.

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1. Introduction

We study the interior Lipschitz regularity for equations of the type

$$F(II, \nu) = 0, \tag{1}$$

where $II = II(S)$ is the second fundamental form of a hypersurface S in R^{n+1} ($n \geq 2$) and $\nu = \nu(S)$ is its normal. We always assume that F is uniformly elliptic in the tangential direction of the surface and Lipschitz in ν with Lipschitz constant linear in $|II|$. See more precise definitions of these terms in the next section. The main goal of this paper is to show that any C^1 viscosity solution has Lipschitz apriori estimates in its interior.

The main theorem of this paper is the following.

Theorem 1 *Suppose F satisfies* (3), (6) *and* (7). *Assume S is, in the sense of* (3), *a C^1 solution of* (1) *in the cylinder $C_1 = B_1 \times [-K, K]$, where $K = [S]_{L^\infty(B_1)} +$*

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$|X_n S(0)| + 1$, in this coordinate system and $X_n(S)$ the X_n coordinate of S such that $(0, X_n(S)) \in S$. Then $S \in Lip(C_{\frac{1}{2}})$. Moreover $\|S\|_{Lip(C_{\frac{1}{2}})}$ has a upper bound depending only on $\|S\|_{L^\infty}$, ellipticity and the dimension. In general, we denote $C_r(x) = B_r(x) \times [-\|S\|_{L^\infty(B_r)} - X_n S(0) - 1, \|S\|_{Lip(B_1)} + X_n S(0) + 1]$.

By the theorems in [1], the Lipschitz regularity of (1) implies $C^{1,\alpha}$ regularity. This kind of estimates is not new for the classical solutions. However, we think our approach gives not only a regularity theory but also a much better geometric intuition of how regularity 'propagates' along a solution surface (see Lemma 5).

Our methods are related with the work of N. Korevaar [2] and its extensions by Korevaar [3], Y. Li [4] and B. Guan and J. Spruck [5], where Lipschitz estimates for C^3 solutions were obtained.

To the contrast, our proof is more natural and geometrical. It is along the line of the regularity theory for free boundaries developed by the first author [6].

In fact, it is one of our current objects to unify the theory of free boundaries and the theory of elliptic equations.

The main technical contribution of this paper is the construction of 'variable' parallel surfaces, which are still subsolutions to the equation of the surface.

Our method also applies to parabolic equations and equations of motions of surfaces by their curvatures.

2. Viscosity Solutions and Preliminary Considerations

Instead of considering the second fundamental form itself, we will deal with its representations in coordinate systems of R^{n+1}.

Definition 1 *Let M be an $(n+1) \times (n+1)$ matrix. If*

$$v^T\ Mv = II(v, v)$$

for

$$v \in TS = \{\nu \cdot v = 0\},$$

we say that M is a representation of II.

Clearly, the representations of a second fundamental form are not unique.

Equation (1) is equivalent to equations of the form,

$$F(M, \nu) = 0 \tag{2}$$

with the condition

$$\mu(e \otimes \nu + \nu \otimes e) + b(\nu \otimes \nu), \nu) = F(M, \nu) \tag{3}$$