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BV SOLUTIONS OF DIRICHLET PROBLEM FOR A CLASS OF DOUBLY NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract The uniqueness and existence of BV solutions to Dirichlet problem of doubly degenerate parabolic equations of the following form

$$\frac{\partial u}{\partial t} = \mathrm{div}(A(|\nabla B(u)|)\nabla B(u)) \quad \text{in } Q_T = \Omega \times (0, T)$$

are studied
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1. Introduction

In this paper, we study Dirichlet problem of the following form

$$\frac{\partial u}{\partial t} = \mathrm{div}(A(|\nabla B(u)|)\nabla B(u)) \quad \text{in } Q_T = \Omega \times (0, T) \tag{1.1}$$

$$u(x, t) = 0 \quad (x, t) \in \partial\Omega \times (0, T) \tag{1.2}$$

$$u(x, 0) = u_0(x) \qquad x \in \Omega \tag{1.3}$$

where $\Omega \subset R^m$ is a bounded region with boundary $\partial\Omega$ appropriately smooth, $A, B \in C^1(R)$ and

$$A(s) = \int_0^s a(\sigma)d\sigma,\ B(s) = \int_0^s b(\sigma)d\sigma,\ a(s) \geq 0,\ b(s) \geq 0,\ b(0) = 0 \tag{1.4}$$

Set $A^i(p) = A(|p|)p_i$, where $p = (p_1, \cdots, p_m) \in R^m$.

We assume

$$0 \leq \frac{\partial A^i(p)}{\partial p_j}\xi_i\xi_j \leq \Lambda|\xi|^2 \qquad \forall \xi \in R^m \tag{1.5}$$

$$\mu_1|p|^q \le A(|p|)|p|^2 \le \mu_2(|p|^q+1), \quad A'(s) \ge 0 \tag{1.6}$$

where $q \ge 2, \Lambda, \mu_1, \mu_2$ are positive constants.

Dirichlet problem (1.1)-(1.3) arises from a variety of diffusion phenomena appeared widely in nature. The Newtonian filtration equations

$$\frac{\partial u}{\partial t} = \Delta\varphi(u), \quad \varphi'(s) \ge 0 \tag{1.7}$$

and non-Newtonian filtration equations

$$\frac{\partial u}{\partial t} = \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) \quad p \ne 2 \tag{1.8}$$

are the special cases of (1.1). Since we only suppose $B'(s) \ge 0$, in general the solution of (1.1)-(1.3) is not continuous and the sense of satisfying the boundary value condition for the solution is also special (see[1]). When $A(s) \equiv 1$, the existences of BV solution to Cauchy problem and Dirichlet problem of (1.1) have been studied(see [2-4]). The existence and uniqueness of solutions with compact support for a class of doubly nonlinear degenerate parabolic equations are considered in [5]. In this note we will investigate the solvability of (1.1)-(1.3) in $BV(Q_T)$ when A(s) satisfies (1.5) and (1.6). In the proof of existence of solution, we use some ideas in [3]. But there are many difficulties to need overcome since (1.1) is degenerate at the points where $A(|\nabla u|) = 0$. The paper is constructed as follows. We first define solutions of the Dirichlet problem (1.1)-(1.3) in Section 2. Subsequently in Section 3 we establish some estimates for the solutions of regularized problem. On the basis of these estimates, we then prove the existence of solutions in Section 4. Section 5 is devoted to study the uniqueness and stability of solution.

2. Main Results

Let Γ_u be the set of all jump points of $u \in BV(Q_T)$, ν be the normal of Γ_u at $X = (x,t), u^+(X)$ and $u^-(X)$ the approximate limits of u at $X \in \Gamma_u$ with respect to $(\nu, Y - X) > 0$ and $(\nu, Y - X) < 0$ respectively(see[6]).

Definition 2.1 *A function $u \in BV(Q_T)\bigcap L^\infty(Q_T)$ is said to be a generalized solution of Dirichlet problem (1.1)-(1.3) if the following conditions are fulfilled*

1. $(B(u))_t \in L^2(Q_T), \quad (B(u))_{x_i} \in L^q(Q_T), \quad i = 1, 2, \cdots, m;$
2. *for almost all* $t \in \Omega \quad \gamma u(x, 0) = u_0(x)$ *where* γu *is the trace of* u;
3. *for almost all* $t \in (0, T) \ \ B(\gamma u) = 0$ *a.e. on* $\partial\Omega$;
4. *u satisfies*

$$\iint_{Q_T} \left\{|u-k|\frac{\partial\varphi_1}{\partial t} - \mathrm{sgn}(u-k)A(|\nabla B(u)|)\nabla B(u)\cdot\nabla\varphi_1\right\}dxdt + \iint_{Q_T} \mathrm{sgn}k\left\{u\frac{\partial\varphi_2}{\partial t} - A(|\nabla B(u)|)\nabla B(u)\cdot\nabla\varphi_2\right\}dxdt \ge 0 \tag{2.1}$$