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EIGENVALUE FUNCTIONS IN EXCITATORY-INHIBITORY NEURONAL NETWORKS*

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Dedicated to Professor Donald Aronson, Professor Boling Guo and Professor Yulin Zhou on the occasion of their birthdays

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Abstract We study the exponential stability of traveling wave solutions of nonlinear systems of integral differential equations arising from nonlinear, nonlocal, synaptically coupled, excitatory-inhibitory neuronal networks. We have proved that exponential stability of traveling waves is equivalent to linear stability. Moreover, if the real parts of nonzero spectrum of an associated linear differential operator have a uniform negative upper bound, namely, $\max\{\mathrm{Re}\lambda\colon\ \lambda \in \sigma(\mathcal{L}),\ \lambda \neq 0\} \leq -D$, for some positive constant D, and $\lambda = 0$ is an algebraically simple eigenvalue of $\mathcal{L}$, then the linear stability follows, where $\mathcal{L}$ is the linear differential operator obtained by linearizing the nonlinear system about its traveling wave and $\sigma(\mathcal{L})$ denotes the spectrum of $\mathcal{L}$. The main aim of this paper is to construct complex analytic functions (also called eigenvalue or Evans functions) for exploring eigenvalues of linear differential operators to study the exponential stability of traveling waves. The zeros of the eigenvalue functions coincide with the eigenvalues of $\mathcal{L}$.

When studying multipulse solutions, some components of the traveling waves cross their thresholds for many times. These crossings cause great difficulty in the construction of the eigenvalue functions. In particular, we have to solve an over-determined system to construct the eigenvalue functions. By investigating asymptotic behaviors as $z \to -\infty$ of candidates for eigenfunctions, we find a way to construct the eigenvalue functions.

By analyzing the zeros of the eigenvalue functions, we can establish the exponential stability of traveling waves arising from neuronal networks.

Key Words Traveling wave solution; exponential stability; linear differential operator; normal spectrum; eigenvalue problem; complex analytic function.

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1. Introduction

Research activities in mathematical biology have become more and more important these days. Many significant results have been obtained by using rigorous mathematical

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analysis. A significant goal of neuroscience is to understand clearly how the nervous system communicates and processes information. The fundamental structural unit of the nervous system is the individual nerve cell which conveys neuronal information through electrical and chemical signals. Patterns of neuronal signals underlie many activities of the brain.These activities include elementary level motor tasks such as walking, breathing and high level cognitive behaviors such as thinking, feeling and leaning. See Professor David Terman [1].

As well known, an important aspect of mathematical neuroscience is to develop and solve mathematical models for neuronal activity patterns. These models are used to understand how various activity patterns are generated and how the patterns change as parameters in the system are modulated. Additionally, the models can also serve to interpret data, test hypotheses, and suggest new experiments, [2]. Since the neuronal systems are typically so complicated, we must be careful to model the system at an appropriate level. Biologically, the model must be complicated enough so that it grasps the main points which are believed to play an important role in the generation of a particular activity pattern. Mathematically, it cannot be so complicated that it is impossible to solve, either analytically or numerically.

Furthermore, a neuronal network's population rhythm results mainly from interactions between three separate components: the intrinsic properties of individual neurons, the synaptic properties of coupling between neurons, and the architecture of coupling (which neurons communicate with each other). These components typically involve a couple of parameters and at least two time scales. The synaptic coupling, for example, can be excitatory or inhibitory, and its possible turn on and turn off rates can vary broadly. Neuronal systems may include several different type of neurons as well as different types of coupling. An important and typically very challenging problem is to determine the role each component plays in shaping the emergent network's behavior. See Terman [1].

In a neuronal network, an individual neuron may, for example, fire repetitive action potentials or bursts of action potentials that are separated by silent phases near quiescent behavior. Examples of population rhythms include synchronized oscillations, in which every cell in the same network fires at the same time and clustering, in which the entire population of neurons breaks up into subpopulations or blocks, every neuron within a single block fires synchronously and different blocks are desynchronized from each other. Of course, much more complicated population rhythms are possible. The activity may, for example, propagate through the network in a pulse-like manner or exhibit chaotic dynamics. See [1, 2]. "

1.1 Traveling wave solutions and stability

Traveling waves are non-constant, bounded, uniformly continuous solutions of nonlinear differential equations. They usually travel at constant speed and their profiles are often time invariant. Stable waves are physically the most interesting solutions. Thus, for a given traveling wave, it is very important to determine its stability relative