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# SELF-SIMILAR SINGULAR SOLUTION OF A P-LAPLACIAN EVOLUTION EQUATION WITH GRADIENT ABSORPTION TERM*

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**Abstract** In this paper we deal with the self-similar singular solution of the p-Laplacian evolution equation $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^q$ for $p > 2$ and $q > 1$ in $R^n \times (0, \infty)$. We prove that when $p > q + n/(n + 1)$ there exist self-similar singular solutions, while $p \leq q+n/(n+1)$ there is no any self-similar singular solution. In case of existence, the self-similar singular solutions are the self-similar very singular solutions, which have compact support. Moreover, the interface relation is obtained.
**Key Words** p-Laplacian evolution equation; gradient absorption; self-similar; singular solution; very singular solution.
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## 1. Introduction and Main Results

In this paper we consider the self-similar singular solution of the p-Laplacian evolution equation with nonlinear gradient absorption term

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u|) - |\nabla u|^q \ \text{ in } \ R^n \times (0, \infty), \tag{1.1}$$

where $p > 2$ and $q > 1$. Here by singular solution we mean a nonnegative and nontrivial solution $u(x, t)$, which is continuous in $R^n \times [0, \infty)\backslash\{(0, 0)\}$ and satisfies

$$\lim_{t\to 0} \sup_{|x|>\varepsilon} u(x, t) = 0, \qquad \forall\ \varepsilon > 0. \tag{1.2}$$

A singular solution $u(x, t)$ is called a very singular solution provided that it satisfies

$$\lim_{t\to 0} \int_{|x|<\varepsilon} u(x, t)dx = \infty, \ \ \forall\ \varepsilon > 0. \tag{1.3}$$

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By self-similar solution we mean that the solution $u(x,t)$ has the following form

$$u(x,t)=(\frac{\alpha}{t})^{\alpha}f(|x|(\frac{\alpha}{t})^{\alpha\beta}),\quad \alpha=\frac{p-q}{2q-p},\quad \beta=\frac{q+1-p}{p-q}. \tag{1.4}$$

To guarantee the constants $\alpha$ and $\beta$ are positive, here we consider the following case

$$2q>p>q,\quad q+1-p>0. \tag{1.5}$$

Consequently, the self-similar singular solution to (1.1), if it exists, satisfies the following ODE boundary problem

$$\begin{cases} (|f'|^{p-2}f')'+\frac{n-1}{r}|f'|^{p-2}f'+\beta rf'+f-|f'|^q=0 \\ f(0)=a>0,\quad \lim_{r\to\infty}r^{1/\beta}f(r)=0, \end{cases} \tag{1.6}$$

where $f=f(r)$ is the function of self-similar variable $r=|x|(\alpha/t)^{\alpha\beta}$, the prime denotes the differentiation with respect to $r$.

In this paper we set

$$\nu=p+(p-2)/\beta=q+(q-1)/\beta>2.$$

Singular solutions were first discovered for the semilinear heat equation

$$u_t=\Delta u-u^p. \tag{1.7}$$

Brezis and Friedman [1] in 1983 proved that (1.7) admits a unique singular solution for every $c\in(0,\infty)$ when $1<p<1+2/n$ such that $\lim_{t\to0}\int_{|x|<\varepsilon}u(x,t)dx=c,\ \ \forall\ \varepsilon>0$, which is called a fundamental solution with initial mass $c$, while it has no such solutions for $p\geq1+2/n$. Shortly, Brezis, Peletier and Terman [2] had proved that (1.7) posses a unique very singular solution when $1<p<1+2/n$. Since that time many authors studied the self-similar singular solutions ( see [3-8] and the references therein) of the following equations

$$u_t=\Delta(u^m)-u^p,\qquad 0<m<\infty,\ \ p>1,$$

$$u_t=\Delta(u^m)-|\nabla u|^p,\qquad 1\leq m<\infty,\ \ p>1,$$

$$u_t=\mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m)-u^q,\qquad 0<m<\infty,\ \ p>1,\ \ q>1.$$

In addition, large time behavior of solutions to the Cauchy problems of the above equations with absorption $u^p$(or $u^q$) can be characterized by their very singular solution, self-similar solutions and fundamental solutions, see [9-14].

To study the boundary value problem (1.6), we consider the initial problem

$$\begin{cases} (|f'|^{p-2}f')'+\frac{n-1}{r}|f'|^{p-2}f'+\beta rf'+f-|f'|^q=0,\quad r>0, \\ f(0)=a>0,\ \ f'(0)=0. \end{cases} \tag{1.8}$$

Let $f(r;a)$ be the solution of (1.8) and $(0,R(a))$ be the maximal existence interval, where $f(r;a)>0$. Our main results read as follows: