J. Partial Diff. Eqs. **16**(2003), 18–28
©International Academic Publishers Vol.16, No.1

HOMOCLINIC ORBIT IN A SIX DIMENSIONAL MODEL OF A PERTURBED HIGHER-ORDER NLS EQUATION

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Dedicated to the 80th birthday of Professor Zhou Yulin

(Received March 12, 2002)

Abstract In this paper, the perturbed higher-order NLS equation with periodic boundary condition is considered. The existence of the homoclinic orbits for the truncation equation is established by Melnikov analysis and geometric singular perturbation theory.

Key Words homoclinic,higher-order NLS equation, perturbation.
2000 MR Subject Classification 35Q53.
Chinese Library Classification O175.29.

1. Introduction

By using the reductive perturbation method, Kodama and Hasegawa proposed a higher-order nonlinear Schrödinger (HNLS) equation

$$\begin{aligned} &iq_t + \tfrac{1}{2}k_1 q_{xx} + \ l\ |q|^2 q \\ &\qquad = -i\varepsilon\left[-\tfrac{1}{6}k_2 q_{xxx} + h_1(q|q|^2)_x - h_2(|q|^2)_x q\right]. \end{aligned} \tag{1}$$

It can be used to describe the propagation of a femtosecond optical pulse in a monomode optical fiber.

In this paper, we consider the following perturbation HNLS equation

$$\begin{aligned} &iu_t + u_{xx} + (|u|^2 - 1)u \\ &\qquad = i\varepsilon\left[\alpha u + \beta_1 u_{xxx} + \beta_2(|u|^2 u)_x + \beta_3(|u|^2)_x u + \Gamma\right] \end{aligned} \tag{2}$$

with periodic boundary condition$u(x+2\pi,t) = u(x,t)$.Where $u = u(x,t)$ is a complex-value function of two real variables t and x, $\alpha, \beta_1, \beta_2, \beta_3$ and Γ are real parameters ($\alpha > 0, \Gamma > 0$), and $\varepsilon > 0$ is a small perturbation parameter. We adopt a three mode Fourier truncation and get a six dimensional ordinary differential equations. This equations will be considered and the persistence of the homoclinic orbits will be obtained by Melnikov's analysis together with the geometrical singular perturbation theory.

2. The Fourier Truncation of the Perturbation HNLS Equation

Suppose that the equation (2) have a solution with the following type

$$u(x,t) = \frac{1}{\sqrt{2}}a(t) + b(t)\cos x + c(t)\sin x. \tag{3}$$

where a, b, and c are complex. Inserting (3) into the perturbed HNLS equation (2) and neglecting the higher Fourier modes yields

$$\begin{aligned}
i\,\dot{a} &+(\frac{1}{2}|a|^2+\frac{1}{2}|b|^2+\frac{1}{2}|c|^2-1)a+\frac{1}{2}(ab^*+a^*b)b+\frac{1}{2}(ac^*+a^*c)c\\
&=i\varepsilon[\alpha a+\frac{1}{2}\beta_3 b(ac^*+a^*c)-\frac{1}{2}\beta_3 c(ab^*+a^*b)+\sqrt{2}\Gamma]\\
i\,\dot{b} &+(\frac{1}{2}|a|^2+\frac{3}{4}|b|^2+\frac{1}{4}|c|^2-2)b+\frac{1}{2}(ab^*+a^*b)a+\frac{1}{4}(bc^*+b^*c)c\\
&=i\varepsilon[\beta_2(\frac{1}{2}|a|^2+\frac{1}{2}|b|^2+\frac{1}{2}|c|^2)c+\frac{1}{2}(\beta_2+\beta_3)(ac^*+a^*c)a\\
&\quad+\frac{1}{4}(\beta_2+2\beta_3)(bc^*+b^*c)b-\frac{1}{4}(\beta_2+2\beta_3)(|b|^2-|c|^2)c]+i\varepsilon(\alpha b-\beta_1 c) \qquad (4)\\
i\,\dot{c} &+(\frac{1}{2}|a|^2+\frac{1}{4}|b|^2+\frac{3}{4}|c|^2-2)c+\frac{1}{2}(ac^*+a^*c)a+\frac{1}{4}(bc^*+b^*c)b\\
&=-i\varepsilon[\beta_2(\frac{1}{2}|a|^2+\frac{1}{2}|b|^2+\frac{1}{2}|c|^2)b+\frac{1}{2}(\beta_2+\beta_3)(ab^*+a^*b)a\\
&\quad+\frac{1}{4}(\beta_2+2\beta_3)(bc^*+b^*c)c+\frac{1}{4}(\beta_2+2\beta_3)(|b|^2-|c|^2)b]+i\varepsilon(\alpha c+\beta_1 b).
\end{aligned}$$

From (4) the unperturbed equations are obtained by setting $\varepsilon = 0$

$$\begin{aligned}
i\,\dot{a} &+(\frac{1}{2}|a|^2+\frac{1}{2}|b|^2+\frac{1}{2}|c|^2-1)a+\frac{1}{2}(ab^*+a^*b)b+\frac{1}{2}(ac^*+a^*c)c=0\\
i\,\dot{b} &+(\frac{1}{2}|a|^2+\frac{3}{4}|b|^2+\frac{1}{4}|c|^2-2)b+\frac{1}{2}(ab^*+a^*b)a+\frac{1}{4}(bc^*+b^*c)c=0 \qquad (5)\\
i\,\dot{c} &+(\frac{1}{2}|a|^2+\frac{1}{4}|b|^2+\frac{3}{4}|c|^2-2)c+\frac{1}{2}(ac^*+a^*c)a+\frac{1}{4}(bc^*+b^*c)b=0.
\end{aligned}$$

By inspection, we see that the unperturbed equations are invariant under the following coordinate transformations

$$(a,b,c)\to(-a,b,c);\,(a,b,c)\to(a,-b,-c). \tag{6a,6b}$$

We want to describe the invariant manifold structure and phase space geometry of (5), we also want ultimately to utilize the generalized Melnikov theory described in [1]. For these purpose, we rewrite the equations (4) in the appropriate form by introducing the following coordinate transformation

$$\begin{aligned}
a &= \rho(t)\exp\{i\theta(t)\}\\
b &= [x_1(t)+ix_2(t)]\exp\{i\theta(t)\} \qquad (7)\\
c &= [y_1(t)+iy_2(t)]\exp\{i\theta(t)\},
\end{aligned}$$