J. Partial Diff. Eqs. **16**(2003), 335–346
 Vol.16 No.4

ON THE GLOBAL ATTRACTOR OF GENERALIZED GINZBURG-LANDAU EQUATION IN ONE SPATIAL DIMENSION

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(Received Nov. 7, 2002)

Abstract The target of this paper is the long time behaviour of solutions for a generalized Ginzburg-Landau equation on $\mathbb{R}$. The authors establish the existence of a global attractor of finite Hausdorff and fractal dimension in a weighted Hilbert space for the equation.

Key Words Global attractor; Hausdorff dimension; Fractal dimension; Generalized Ginzburg-Landau equation.

2000 MR Subject Classification 35Q35, 35Q80.

Chinese Library Classification O175.24.

1. Introduction

The generalized Ginzburg-Landau (GGL) equation in one spatial dimension is the equation of the following form

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^{2\sigma} u,$$

where $\alpha_k = a_k + ib_k (k = 0, 1, 2, 3, 4, 5)$ are complex constants and σ is a positive constant.

The GGL equation arises as a generic amplitude equation close to the onset of instabilities that lead to chaotic dynamics in fluid mechanical systems, as well as in reaction-diffusion processes, nonliear optics, electric foring of liquid crystal and many other fields. For the further background of the equation, the reader may consult Doelman [1], Duan [2-4], Eckhaus and Iooss [5].

For long time behaviours of solutions, Guo and Gao [6] and Guo and Wang [7] have proved the existence of global attractors of the Cauchy problem of GGL equation with periodic boundary condition for the cases of one and two spatial dimensions. They have also discussed Gevrey regularity and approximate inertial manifold for the same problem in two spatial dimensions [8]. Guo and Li [9] have showed the existence of a global attractor in a so-called uniformly weighted space for Cauchy problem of GGL equation in an unbounded two-dimensional-domain, in which $\alpha_0 = a_0 > 0$ and $\operatorname{Re}\alpha_2 = 0$ the so-called super-critical case was considered and the weight function was integrable.

In this paper we are interested in Cauchy problem of the following GGL equation

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^4 u + f, \tag{1}$$

with initial data

$$u|_{t=0} = u^0, \tag{2}$$

where $\alpha_0 = a_0 < 0$, $\operatorname{Re}\alpha_2 > 0$, that is so-called sub-critical case, and $f = f(x)$ is known.

Under the condition $a_0 < 0$, the dynamics of GGL is dissipative, while the compactness of the solution operator to the GGL is absent due to the noncompact embedding $H^1 \hookrightarrow L^2$ in $I\!R$. To obtain certain attracting behavior for Cauchy problem (1) and (2) in some sense, we study the problem in the context of weighted Hilbert spaces $H_{0,r}$ $(r > 0)$ with the norm

$$\|u\|_{0,r} = \int_{-\infty}^{+\infty} |u(x)|^2 (1 + x^2)^r dx.$$

We will also employ weighted Hilbert spaces $H_{l,r}$ whose norm is

$$\|u\|_{l,r}^2 = \sum_{j=0}^{l} \|\partial_x^j u\|_{0,r}^2.$$

Such weighted spaces were used in [10] for Cauchy problem of certain reaction-diffusion equation.

This work is organized as follows: the weighted estimates are presented in the second section, the discussion follows Ref. [11]. The existence and uniqueness of the global solution in the weighted space are shown in the third section by Henry's theorem. In the fourth section the existence of the global attractor is proved in an analogous way as Ref. [10]. Finally the finite Hausdorff and fractal dimensions of the attractor are estimated in the last section.

2. Weighted Estimates of Solutions

In this section a uniform bound in time t for the solutions of (1) and (2) in weighted space $H_{l,r}$ will be figured out, which is crucial for global existence of a solution and the attracting behaviors of the solution.

Before stating the results, we need to prepare some technical tools. Let

$$\varphi(x) = \varphi_\epsilon(x) = (1 + (\epsilon x)^2)^r$$