J. Partial Diff. Eqs. 14(2001), 111–116
©International Academic Publishers
Vol.14, No.2

POSITIVE SOLUTIONS OF FULLY NONLINEAR ELLIPTIC EQUATIONS ON GENERAL BOUNDED DOMAINS*

Li Meisheng
(Department of Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China)
Bao Jiguang
(Department of Mathematics, Beijing Normal University, Beijing 100875, China)
(Received May 16, 2000; revised Aug. 11, 2000)

Abstract We prove the refined ABP maximum principle, comparison principle, and related existence and uniqueness theorem for the positive solutions of the Dirichlet problems of second order fully nonlinear elliptic equations on arbitrary bounded domains.

Key Words Refined ABP maximum principle; fully nonlinear equation; arbitrary bounded domain.

1991 MR Subject Classification 35J60.

Chinese Library Classification O175.29.

1. Introduction

This paper is concerned with positive solutions of secoind order fully nonlinear elliptic equations of the form

$$F(D^2u, x) = f(u, x)$$

in a general bounded domain $\Omega \subset \boldsymbol{R}^n$. Fully nonlinear elliptic equations in regular domains have been treated under the natural structure conditions by several authors, notably Evans [1], Trudinger [2] and so on. However, there seem to be few results in general bounded domains. For an irregular bounded domain Ω, the boundary condition $u = 0$ on $\partial\Omega$ is too strong a hypothesis. Generally, one cannot prescribe boundary values of solutions at every point of $\partial\Omega$. In such case, it is difficult to estimate bounds of the solutions for the usual maximum principles are invalid. But, Berestycki, Nirenberg and Varadhan [3] worked with a refined version for the linear elliptic equations in general bounded domains.

* This work supported by National Natural Science Foundation of China (Grant No.19771009) and the Doctoral Program Foundation of Education Ministry of China (Grant No.1999002705).

Similar to that paper, a function u_0 is introduced, which is the limit, in some sense, of the function sequence $\{u_k\}$ defined as follows: let $\{\Omega_k\}$ be a sequence of smooth domains satisfying

$$\Omega_k \subset \bar{\Omega}_k \subset \Omega_{k+1}, \quad k = 1, 2, \cdots, \quad \bigcup_{k=1}^{\infty} \Omega_k = \Omega$$

We always assume that $F \in C^2(R^{n\times n} \times \Omega)$ satisfies the structure conditions:

(F1) $\lambda|\xi|^2 \le F^{ij}\xi_i\xi_j \le \lambda^{-1}|\xi|^2$,

(F2) $|F_x|, |F_{xx}| \le \mu(1+|r|), \quad |F_{rx}| \le \mu$,

(F3) $F_{rr} \le 0$,

for all $(r,x) \in R^{n\times n} \times \Omega$, where λ, μ are positive constants, and $F^{ij}(r,x) = F_{r_{ij}}(r,x)$. By the standard existence theory for classical solutions [4, Theorem 6.14], there is a unique solution $u_k \in C^{2,\alpha}(\bar{\Omega}_k)$ of

$$F^{ij}(0,x)D_{ij}u_k = -1, \quad \text{in } \Omega_k \tag{1}$$

$$u_k = 0, \quad \text{on } \partial\Omega_k \tag{2}$$

where $\alpha \in (0,1)$. By the usual maximum principle and the interior Schauder estimates [4, Theorem 3.7 and Theorem 6.2],

$$0 < u_k(x) \le C, \quad x \in \Omega_k$$

$$\|u_k\|_{C^{2,\alpha}(\Omega_k)} \le C$$

where the constant C and $\alpha \in (0,1)$ do not depend on k. Using the maximum principle again we find that the sequence $\{u_k(x)\}$ is strickly monotone increasing in k. Consequently, $u_k \to u_0$ in $C^2(\Omega)$, and $u_0 \in C^{2,\alpha}(\Omega) \cap L^\infty(\Omega)$ satisfies

$$F^{ij}(0,x)D_{ij}u_0 = -1, \quad u_0 > 0, \quad \text{in } \Omega \tag{3}$$

It is easily seen that the function u_0 defined above is independent of the choices of the subdomains Ω_k.

The following notation will be used. For a sequence $x^l \to \partial\Omega$, we say $x^l \xrightarrow{u_0} \partial\Omega$ if $u_0(x^l) \to 0$. Given $u \in C(\Omega)$, the notation $u \overset{u_0}{=} 0$ on $\partial\Omega$ means: along any sequence $x^l \xrightarrow{u_0} \partial\Omega$, we have $u(x^l) \to 0$. For a regular domain Ω, it is the same as $u(x) = 0$ for $x \in \partial\Omega$.

In the above sense, the boundary condition

$$\lim_{x\to\partial\Omega} \sup u(x) \le 0$$

is replaced by

$$\lim_{l\to\infty} \sup u(x^l) \le 0, \text{ if } x^l \xrightarrow{u_0} \partial\Omega$$