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MOSER-TRÜDINGER INEQUALITY ON COMPACT RIEMANNIAN MANIFOLDS OF DIMENSION TWO

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Abstract In this paper, we prove Moser-Trüdinger inequality in any two dimensional manifolds. Let (M, g_M) be a two dimensional manifold without boundary and (g, g_N) with boundary, we shall prove the following three inequalities:

$$\sup_{u\in H^1(M),\text{ and }\|u\|_{H^1(M)}=1}\int_M e^{4\pi u^2}<+\infty$$

$$\sup_{u\in H^1(M),\int_M u=0,\text{ and }\int_M|\nabla u|^2=1}\int_M e^{4\pi u^2}<+\infty$$

$$\sup_{u\in H^1_0(N),\text{ and }\int_M|\nabla u|^2=1}\int_M e^{4\pi u^2}<+\infty$$

Moreover, we shall show that there exist of extremal functions which attain the above three inequalities.
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1. Introduction

Let (M, g) be a compact Riemannian manifold of dimension n, and let $H^{1,p}$ denote the completion of $C^\infty(M)$ in the norm $\|u\|^p_{H^{1,p}}=\int_M(|\nabla u|^p+|u|^p)dV_g$. It is well known that the Orlicz space embedding

$$H^{1,n}(M)\ni u\to e^{|u|^{\frac{n}{n-1}}}\in L^p\text{ for any }p$$

is well-defined and smooth locally. From an early result in [1], it is known that $\int_M e^{\alpha|u|^{\frac{n}{n-1}}}$ is bounded for some positive number α, whenever $\|u\|_{H^{1,n}}\le 1$. The limiting behavior of exponent $\alpha_n=n(\omega_{n-1})^{\frac{1}{n-1}}$ is much interesting, where ω_{n-1} denotes the

$(n-1)$-dimensional measure of the unit sphere $S^{n-1} \subset R^n$. Many results for this problem have been obtained in the case that M is a bounded subdomain in R^n, especially when $n = 2$.

Let Ω be a bounded domain in R^n, Moser obtained the following famous inequality in [2]:

Theorem A (Moser) *There exists a constant c_n, depending only on the dimension n such that for all $\alpha \le \alpha_n$,*

$$\int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx \le c_n \tag{1.1}$$

for all $u \in H_0^{1,n}(\Omega)$ with $\|u\|_{H^{1,n}} \le 1$.

Moreover, for $\alpha > \alpha_n$,

$$\sup_{\|\nabla u\|_{L^2(\Omega)}=1, u \in H_0^1(\Omega)} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx = +\infty$$

About fifteen years later, Lennart Carleson and Sun-Yung A.Chang proved the existence of extremal function which attains the supremum in (1.1), provided Ω was the unit ball in R^n([3]). If such a function does not exist, they obtain the following estimation

$$\sup_{\int_\Omega |\nabla u|^2 = 1, u|_{\partial\Omega}=0} \frac{1}{|\Omega|} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}} = 1 + e^{1+1/2+\cdots+1/(n-1)}$$

Then they succeed in constructing a comparison function which contradicts the above identity.

Michael Struwe studied the existence of extremal functions for nonsymmetric domain. In the case that $n = 2$, he obtained a sufficient condition for the existence, using the blow-up analysis(cf[4]). In 1992, Martin Flucher introduced another method, the comformal rearrangement, and derive an isoperimentric inequality, which implies the existence of extremal functions when Ω is a bounded domain in R^2 (cf[5]).

In this paper, we shall apply blow-up analysis to study the Moser-Trüdinger inequality on manifolds. First, we shall show it is still true on 2 dimensional Riemannian manifold, and then give a more general result similar to the estimation obtained by L.Carleson and S.Y.A.Chang when $n = 2$. In the end we shall prove that the extremal function generally exists, that is to say, the existence is independent of the differential structure, the metric and then the curvature of the manifold.

First result in this direction was also obtained by Moser, who proved the following theorem in [2]:

Theorem B (Moser) *If u is a smooth function defined on S^2, satisfying*

$$\int_{S^2} |\nabla u|^2 \le 1, \quad \int_{S^2} u = 0$$