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# LOCAL CLASSICAL SOLUTIONS TO THE EQUATIONS OF RELATIVISTIC HYDRODYNAMICS

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**Abstract** In this paper, we prove that the convexity of the negative thermodynamical entropy of the equations of relativistic hydrodynamics for ideal gas keeps its invariance under the Lorentz transformation if and only if the local sound speed is less than the light speed in vacuum. Then a symmetric form for the equations of relativistic hydrodynamics is presented and the local classical solution is obtained. Based on this, we prove that the nonrelativistic limit of the local classical solution to the relativistic hydrodynamics equations for relativistic gas is the local classical solution of the Euler equations for polytropic gas.
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## 1. Introduction

The momentum-energy tensor [1,2] $T^{\alpha\beta}(\alpha,\beta=0,1,2,3)$ of relativistic hydrodynamics satisfies

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta}=0,\quad \alpha,\beta=0,1,2,3 \tag{1.1}$$

When $n_0$ is the density of particle in the instantaneous inertial frame and $m_0$ the mass of every static particle of the gas, $\rho_0=n_0m_0$. $\rho_0$ satisfies

$$\frac{\partial \rho_0 u^\beta}{\partial x^\beta}=0,\quad \beta=0,1,2,3 \tag{1.2}$$

where

$$T^{\alpha\beta}=\frac{1}{c^2}(\mu+p_0)u^{\alpha\beta}+p_0g^{\alpha\beta}(\alpha,\beta=0,1,2,3),x^0=ct,x^i=x_i,\ i=1,2,3$$

with $c$ the light speed, $g^{\alpha\beta}(\alpha,\beta=0,1,2,3)$ the Minkovski metric tensor, $p_0$ the pressure in the instantaneous inertial frame, and $u^\alpha(\alpha=0,1,2,3)$ the four-dimensional velocity in Minkovski space,

$$(u^0,u^1,u^2,u^3)=\gamma(c,v_1,v_2,v_3),v=|v|$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$$

and

$$\mu = \rho_0 c^2 + \rho_0 e_0$$

where $e_0$ is the internal energy in the instantaneous inertial frame and $v$ is the velocity, and

$$e_0 = e_0(p_0, \rho_0)$$

It is well known that there exists an additional conservation law on the thermodynamical entropy $s_0$[1, 2]

$$\frac{\partial \rho_0 u^\beta s_0}{\partial x^\beta} = 0, \quad \beta = 0, 1, 2, 3 \tag{1.3}$$

and we can rewrite (1.1) and (1.2) in the form of symmetric hyperbolic conservation laws with the method of Godounov [3,4] if we can show the strict convexity of $-\rho_0 u_0 s_0$.

Below we will find that this work is more complicate in the relativistic case than that the classical one. In fact, if we assume that in the instantaneous inertial frame the internal energy is a strictly convex function of the specific volume and entropy, then for classical Euler equations, we can show the strict convexity of $-\rho_0 s_0$ in the laboratorial frame. While for the equations of relativistic hydrodynamics, we can not ensure that the convexity of $-\rho_0 u_0 s_0$ keeps its invariance under Lorentz transformation. This convexity is of no use in the relativistic case indeed.

In his paper [5], Makino considered the ultra-relativistic gas which satisfies

$$p_0 = a\mu, \quad 0 < a < 1$$

with $a$ a constant. For the gas, we can not get an additional conservation law on entropy from (1.1) and (1.2) by the thermodynamical laws, because for the gas, the entropy can not be regarded as a thermodynamical variable. By employing of the equation of entropy-entropy flux, he presented a convex entropy for ultra-relativistic gas. Of course, this entropy has nothing to do with the thermodynamical entropy. Based on this, Makino [5] obtained the local smooth solutions to the equations of relativistic hydrodynamics by the theorem of local existence of the classical solution for the equations of symmetric hyperbolic equations of order one and gave the nonrelativistic limits of the solution of the equations of relativistic hydrodynamics.

In this paper, we repeat that work for ideal gas. Obviously, the most important work is to find out the condition under which $-\rho_0 u_0 s_0$ is a strictly convex function of $T^{\alpha 0} (\alpha = 0, 1, 2, 3)$ and $\rho_0 u^0$ and the convexity keep its invariance under Lorentz transformation.

## 2. Strictly Convex Entropy

At first, we set that those thermodynamical variables in the instantaneous inertial frame have the subscript 0, and those in the laboratorial frame haven't it.