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MEAN VALUE THEOREM AND POHOZAEV TYPE IDENTITIES OF GENERALIZED GREINER OPERATOR AND THEIR APPLICATIONS*

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Abstract In this paper, a fundamental solution at the origin and mean value theorem of generalized Greiner operator are given. Then the Hardy type inequality and some Pohozaev type identities are proved. As their applications, some nonexistence results of semilinear nonelliptic equation and unique continuation are discussed.
Key Words Fundamental solution; mean value theorem; Hardy type inequality; Pohozaev type identity; nonexistence; unique continuation.
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1. Introduction

In recent decades the study of properties of the kohn-Laplacian Δ_H (which satisfy the Hörmander's condition of hypoellipticity see [1]) on the Heisenberg group H^n has made great advances. Folland has given the fundamental solution of $-\Delta_H$ with singularity at the origin, see [2]. Garofalo and Lanconelli gave Hardy type inequality for Δ_H on the base of the work of Folland, and considered the unique continuation see [3]. In [4], Pohozaev type identity on Δ_H was deduced and used to obtain the nonexistence of semilinear subelliptic equation. For generalization see [5].

The aim of this paper is to give some interesting properties of generalized Greiner operator:

$$L = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) \tag{1.1}$$

where $X_j = \dfrac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\dfrac{\partial}{\partial t}, Y_j = \dfrac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\dfrac{\partial}{\partial t}, z_j = x_j + \sqrt{-1}y_j, j = 1,\cdots,n, z = (z_1,\cdots,z_n) \in C^n, t \in R, k \geq 1$. We note that the nonelliptic operator L becomes Δ_H in the case $k = 1$, and when $k = 2, 3, \cdots, L$ is first introduced by Greiner in the study of the boundary Cauchy-Riemann complex, see [6]. It is well-known that

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if $k > 1$, the vector fields $\{X_j, Y_j\}$ are not the basis for any Lie group and L does not possess the translation invariance. But clearly L is a quasihomogeneous PDOs ([7,8]). We will establish the fundamental solution at the origin of Greiner operator, Hardy type inequality and Pohozaev type identity associated to the vector fields $\{X_j, Y_j\}_{\{j=1,\cdots n\}}$. To do so, we need to give some preliminary results on the operator L and vector fields $\{X_j, Y_j\}$ which generalize ones on $\triangle_H$.

The generalized gradient of L is defined as

$$\nabla_L = (X_1, \cdots, X_n, Y_1, \cdots, Y_n) \tag{1.2}$$

A natural family of anisotropic dilations attached to L is given by

$$\delta_r(z,t) = (rz, r^{2k}t), \quad r > 0 \tag{1.3}$$

It is easy to see that

$$L(\delta_r u) = r^2 \delta_r(Lu) \tag{1.4}$$

i.e., L is homogeneous of degree two with respect to δ_r.

Consider the smooth vector field on R^{2n+1}

$$X = \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j} \right) + 2kt \frac{\partial}{\partial t} \tag{1.5}$$

It is readily verified that X is the generator of the group $\{\delta_r\}_{r>0}$.

Denote by A a $(2n+1) \times (2n+1)$ symmetrical matrix, whose elements are: $a_{ij} = \delta_{ij}, i, j = 1, \cdots, 2n; a_{2n+1,j} = 2ky_j|z|^{2k-2}, j = 1, \cdots, n; a_{2n+1,n+j} = -2kx_j|z|^{2k-2}, j = 1, \cdots, n; a_{2n+1,2n+1} = 4k^2|z|^{4k-2}$, then L can be represented as

$$L = \operatorname{div}(A\nabla) \tag{1.6}$$

where ∇ means the Euclidean gradient.

It is easy to check that $A\nabla u \cdot \nabla v = \nabla_L u \cdot \nabla_L v$, therefore

$$A\nabla u \cdot \nabla v = \nabla u A \nabla v \tag{1.7}$$

Note that

$$d(\delta_r(z,t)) = r^Q dzdt$$

where $dzdt$ denotes the Lebesgue measure on R^{2n+1} and

$$Q = 2n + 2k \tag{1.8}$$

The distance function related to $\{X_j, Y_j\}$ is given by

$$d(z,t) = (|z|^{4k} + t^2)^{\frac{1}{4k}} \tag{1.9}$$