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FINITE SPEED OF PROPAGATION OF PERTURBATIONS FOR THE CAHN-HILLIARD EQUATION WITH DEGENERATE MOBILITY*

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Abstract This paper is devoted to the Cahn-Hilliard equation with degenerate mobility in two spatial variables with a typical case modelling thin viscous film spreading over a solid surface. We establish the existence of radial symmetric solutions with the property of finite speed of perturbations.

Key Words Cahn-Hilliard equation; degenerate mobility; finite speed of propagation.

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1. Introduction

In this paper we study the existence of radial symmetric solutions with the property of finite speed of perturbations of the Cahn-Hilliard equation with degenerate mobility in two spatial variables

$$\frac{\partial u}{\partial t} + \operatorname{div}\left[|u|^n(k\nabla\Delta u - \nabla A(u))\right] = 0$$

where $n > 0$, $A(s)$ is appropriately smooth and satisfies the following structure condition

$$H(s) \equiv \int_0^s A(\sigma)d\sigma \geq -\mu_1, \quad |A'(s)| \leq \mu_2|s|^2 + \mu_3 \tag{H}$$

for some positive constants μ_1, μ_2 and μ_3. A reasonable choice of $A(s)$ is the cubic polynomial, namely

$$A(s) = \gamma_1 s^3 + \gamma_2 s^2 + \gamma_3 s + \gamma_4, \quad \gamma_1 > 0$$

which corresponds to the so called double-well potential

$$H(s) = \frac{1}{4}\gamma_1 s^4 + \frac{1}{3}\gamma_2 s^3 + \frac{1}{2}\gamma_3 s^2 + \gamma_4 s$$

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The Cahn-Hilliard equation was introduced to study several diffusion processes, such as phase separation in binary alloys, see for example [1], [2]. In particular, the two-dimensional case can be used as a mathematical model describing the lubrication for thin viscous films and spreading droplets over a solid surface as well as the flow of a thin neck of fluid in a Hele-Shaw cell, see [3]–[6]. For several past years, to the Cahn-Hilliard equations much attention has been paid, and there are many contributions that are devoted to the equation with constant mobility. As for the equation with degenerate mobility, we refer to [7]–[10] for the one-dimensional case. It was Elliott and Garcke [11] who first established the basic existence results of weak solutions for multi-spatial variables, see also [12], [13]. It is much interesting to discuss the properties of solutions. In this respect, we have discussed the nonnegativity of weak solutions in our recent work [14].

This paper is a natural continuation of the investigation of properties of solutions. Our purpose is to seek the solutions with the property of finite speed of propagation of perturbations of the two-dimensional Cahn-Hilliard equation. What we are looking for is the solutions with radial symmetric property, namely, the solution u satisfying $u(x,t)=u(|x|,t)$. Let B_R be the ball centers at the origin with radius R, and introduce the radial variable $r=|x|$. Then we see that the radial symmetric solution satisfies

$$\begin{aligned}&\frac{\partial(ru)}{\partial t}+\frac{\partial}{\partial r}\left\{r|u|^n\left[k\frac{\partial V}{\partial r}-A'(u)\frac{\partial u}{\partial r}\right]\right\}=0\\&rV=\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right),\quad (r,t)\in Q_T\equiv(0,R)\times(0,T)\end{aligned}\tag{1.1}$$

which is degenerate at the points whenever $r=0$ or $u=0$. We consider the boundary value problem for the equation (1.1) with the boundary value conditions

$$\left.\frac{\partial u}{\partial r}\right|_{r=0}=\left.\frac{\partial u}{\partial r}\right|_{r=R}=J|_{r=0}=J|_{r=R}=0\tag{1.2}$$

and the initial value condition

$$u(r,0)=u_0(r),\quad r\in(0,R)\tag{1.3}$$

where

$$J=r|u|^n\left[k\frac{\partial V}{\partial r}-A'(u)\frac{\partial u}{\partial r}\right]$$

Because of the degeneracy, the problem (1.1)–(1.3) does not admit classical solutions in general. So, we introduce weak solutions in the sense of the following.

Definition *A function u is said to be a weak solution of the problem* (1.1)–(1.3), *if the following conditions are fulfilled:*

1) *$ru(r,t)$ is continuous in $\bar{Q}_T$;*

2) $\sqrt{r|u|^n}\dfrac{\partial u}{\partial r}, \sqrt{r|u|^n}\dfrac{\partial^2 u}{\partial r^2}, \sqrt{r|u|^n}\dfrac{\partial^3 u}{\partial r^3}\in L^2(P)$, *where* $P=Q_T\backslash\{(r,t);u(r,t)=0\}$;