INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 11, Number 4, Pages 726–744

NUMERICAL SCHEMES FOR MULTI PHASE QUADRATURE DOMAINS

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Abstract. In this work, numerical schemes to approximate the solution of one and multi phase quadrature domains are presented. We shall construct a monotone, stable and consistent finite difference method for both one and two phase cases, which converges to the viscosity solution of the partial differential equation arising from the corresponding quadrature domain theory. Moreover, we will discuss the numerical implementation of the resulting approach and present computational tests.

Key words. Quadrature domain, Free boundary problem, Finite difference method, Degenerate elliptic equation.

1. Preliminaries

The subject of the quadrature domains, QDs, has been extensively studied over the last half-century and most of the papers deal with the one phase case, e.g., see [10], [12] and [17]. There is a wide range of applications of quadrature domains in physical problems. For instance, Richardson in [19] has studied the Hele-Shaw problem involving a moving boundary problem by driving a flow between two parallel planes without considering surface tension. He opened a crucial and new theory which now is a well developed subject. The solution of the Hele-Shaw problem can be figure out as a one phase quadrature domain.

To the best of our knowledge, most of the authors have studied theoretical aspects of this field and there is a few literature on numerical approach to the quadrature domains. The authors have presented some numerical schemes to approach the one phase quadrature domain in [6]. The main contribution of this paper is to investigate different numerical approximations for the one, two and multi phase quadrature domains.

The outline of this paper is as follows

- We will state the problem in Section two and provide the explanation of the one and the two phase cases and the corresponding partial differential equations, PDEs.
- Section three consists of an introduction to the degenerate elliptic equation and the viscosity solutions.
- Section four is devoted to reformulate the problem for the one and the two phase case. We provide two degenerate elliptic equations and investigate the relation between their viscosity solutions and the weak solutions of the PDEs.

Received by the editors November 11, 2012 and, in revised form, February 2, 2014.

2000 *Mathematics Subject Classification.* 35J70, 65M06.

F. Bozorgnia was supported by the UT Austin-Portugal partnership through the FCT post-doctoral fellowship SFRH/BPD/33962/2009 and grants PTDC/MAT/114397/2009, UT Austin/MAT/0057/2008.

- In Section five we discretize the reformulated problems and introduce our numerical algorithms based on finite difference method. Through this section we concentrate on a special measure, the Dirac measure, and explain the schemes for this case.
- In the last section we shall examine the algorithms by studying some numerical examples.

2. Problem Setting

Let μ_i, $i = 1, \cdots, m$ be given finite measures with compact supports and $\lambda_i(x)$ be non-negative Lipschitz continuous functions. In this article, we investigate the following problem.

Problem: Find functions u_i and domains $\Omega_i := \{x \in \mathbb{R}^N | u_i(x) > 0\}$ for $i = 1, \cdots, m$ such that $\operatorname{supp}(\mu_i) \subset \Omega_i$ and

$$
\begin{cases}
\Delta u_i = \lambda_i \chi_{\Omega_i} - \mu_i & \text{in } \mathbb{R}^N, \\
u_i = 0 & \text{in } \Omega_j, j \neq i, \\
|\nabla u_i| = |\nabla u_j| & \text{on } \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j, \\
|\nabla u_i| = 0 & \text{on } \partial\Omega_i \setminus \bigcup \Gamma_{ij},
\end{cases}
\tag{2.1}
$$

which is understood in the distribution sense. For an illustration of the problem

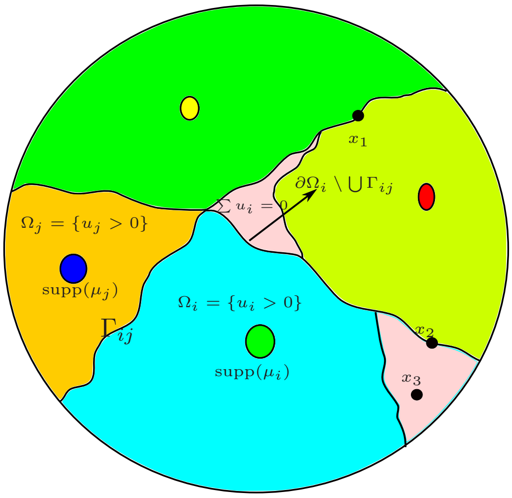

FIGURE 1. This figure shows the supports of the measures and supports of the solution of (2.1) and the corresponding free boundary. The points x_1, x_2 and x_3 are examples of points with different multiplicity, see definition 2.2.

see Figure 1. This problem is related to geometric flows and integral identity in potential theory.

The main contribution of this paper is to construct a finite difference method to approximate the solution of (2.1). We also prove that the numerical approximation converges to the viscosity solution of problem (2.1) for the cases $m = 1$ and $m = 2$. These cases arise from the quadrature domains theory which is quite well studied for the one phase case, see for instance [12] and [17] and the references therein.