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THE HAMILTONIAN SYSTEMS OF THE LCZ HIERARCHY BY NONLINEARIZATION

Li Lu
(Department of Physics, Shanxi University, Taiyuan 030006, China)
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Abstract In this paper, we first search for the Hamiltonian structure of LCZ hierarchy by use of a trace identity. Then we determine a higher-order constraint condition between the potentials and the eigenfunctions of the LCZ spectral problem, and under this constraint condition, the Lax pairs of LCZ hierarchy are all nonlinearized into the finite-dimensional integrable Hamiltonian systems in Liouville sense.
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1. Introduction

It is well known that finding new finite-dimensional completely integrable Hamiltonian systems in Liouville sense is very important [1]. Cao [2] devoloped an approach to produce the finite-dimensional integrable systems for AKNS hierarchy by the nonlinearization of Lax pair of evolution equations under certain constraints between the potentials and the eigenfunctions. Recently, on the basis of Cao's work, Zeng-Li [3] proposed the so-called higher-order symmetric constraint to get the finite-dimensional integrable Hamiltonian systems. According to this approach, many finite-dimensional integrable Hamiltonian systems are obtained [4–6].

For the following LCZ spectral problem

$$\Phi_x = \begin{pmatrix} -i\lambda + r & q + r \\ q - r & i\lambda - r \end{pmatrix} \Phi \tag{1}$$

Qiao [7], Mu [8] et al. presented respectively the commutator representation and a complete integrable Hamiltonian system in the Liouville sense under Bargmann constraint condition. In this paper, the Hamiltonian structure of the LCZ hierarchy is given by using a trace identity, and the so-called higher-order constraint condition between the potentials and the eigenfunctions is obtained by the Hamiltonian structure. Under this constraint, a completely integrable Hamiltonian system is obtained.

The layout of this paper is as follows. In Section 2, we will give the LCZ integrable hierarchy by zero-curvature representation and search for its Hamiltonian structure by use of a trace identity. Then, in Section 3, we determine a higher-order constraint between the potentials and the eigenfunctions of LCZ spectral problem, and under this constraint, the Lax pair of LCZ hierarchy are all nonlinearized into finite-dimensional integrable Hamiltonian systems in Liouville sense.

2. LCZ Hierarchy and Its Hamiltonian Structure

We consider LCZ spectral problem

$$\Phi_x = U(u,\lambda)\Phi, \quad U(u,\lambda) = \begin{pmatrix} -i\lambda + r & q+r \\ q-r & i\lambda - r \end{pmatrix} \tag{2}$$

where $u = (q,r)^T$ is the potential function, λ is the spectral parameter, $-i^2 = 1$. In order to derive LCZ hierarchy of evolution equation by using zero-curvature equation, we first solve the following adjoint representation equation of (2)

$$V_x = [U, V] \tag{3}$$

Let us choose

$$V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{m=0}^{\infty} V_m \lambda^{-m}, \; V_m = \begin{pmatrix} a_m & b_m \\ c_m & -a_m \end{pmatrix}$$

and on setting $a = \sum_{m=0}^{\infty} a_m \lambda^{-m}$, $b = \sum_{m=0}^{\infty} b_m \lambda^{-m}$, $c = \sum_{m=0}^{\infty} c_m \lambda^{-m}$, from (3) we can obtain the following recursion relations to determine a_m, b_m, c_m

$$\begin{cases} b_0 = c_0 = 0 \\ a_m = \partial^{-1} r(b_m + c_m) - \partial^{-1} q(b_m - c_m), & m \geq 0 \\ b_{m+1} = i(q+r)a_m - irb_m + \frac{1}{2} i\partial b_m, & m \geq 0 \\ c_{m+1} = i(q-r)a_m - irc_m - \frac{1}{2} i\partial c_m, & m \geq 0 \end{cases} \tag{4}$$

in which we choose $a_0 = 1$ and assume that $a_m|_{u=0} = b_m|_{u=0} = c_m|_{u=0} = 0$ $(m \geq 1)$, which means to select constants of integration to be zero when $m \geq 1$. In this way, the recursion relations (4) uniquely determine a series of polynomial functions with respect to $u, u_x, u_{xx}, \cdots$. By using (4), for example, we can work out

$$b_0 = c_0 = 0, \quad a_0 = 1$$

$$b_1 = i(q+r), \quad c_1 = i(q-r), \quad a_1 = 0$$

$$b_2 = r(q+r) - \frac{1}{2}(q+r)_x, c_2 = r(q-r) + \frac{1}{2}(q-r)_x, \; a_2 = \frac{1}{2}(q^2 - r^2)$$