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STABILITY AND HOPF BIFURCATION OF STATIONARY SOLUTION OF A DELAY EQUATION*

Zhou Li
(Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China)

Fu Yiping
(Department of Mathematics, South China University of Technology, Guangzhou 510641, China)

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Abstract In this paper we investigate a Logistic equation with delay and it is shown that if $b(x) > c(x)$, the stationary solution is globally asymptotically stable; if τ is small, $U(x)$ is locally stable; if $b(x) < c(x)$, there is Hopf bifurcation from $U(x)$.

Key Words Logistic equation; delay; stability; Hopf bifurcation.
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1. Introduction

In this paper we study the following Logistic equation with instantaneous and delay effects

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(x,t)[a(x) - b(x)u(x,t) - c(x)u(x,t-\tau)], && \Omega \times [0,\infty) \\
&B[u](x,t) = 0, && \partial\Omega \times [0,\infty) \\
&u(x,t) = \eta(x,t), && \Omega \times [-\tau, 0]
\end{aligned} \tag{1.1}$$

where $\eta \in C([-\tau,0], H_0^1[0,\pi])$, $\tau \geq 0$ is constant. The functions $a(x), b(x), c(x)$ are positive and Hölder continuous on $\overline{\Omega}$. The boundary condition is given by $Bu = u$ or $Bu = \dfrac{\partial u}{\partial n} + \gamma(x)u$ where $\gamma \in C^{1+\alpha}(\partial\Omega)$, $\gamma(x) \geq 0$ on $\partial\Omega$ and $\dfrac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$.

The problem (1.1) describes the evolution of population u subject to diffusion, having delay effects in the growth rate. The related problems when a, b, c are constants

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or related ordinary differential equation have been treated extensively [1–5, and their references].

It is well known that the steady-state problem

$$\frac{\partial^2 u}{\partial x^2} + u(a(x) - b(x)u - c(x)u) = 0, \quad x \in \Omega$$
$$Bu = 0, \quad x \in \partial\Omega$$

(1) if $\lambda_1 \geq 1$, it has only the trivial solution 0 which is globally asymptotically stable with respect to every nonnegative initial function,

(2) if $\lambda_1 < 1$, it has a unique positive solution $U(x)$ which is globally asymptotically stable with respect to every nonnegative, nontrivial initial function,

where λ_1 is the smallest eigenvalue of the eigenvalue problem

$$\frac{\partial^2 \phi}{\partial x^2} + \lambda a(x)\phi = 0, \quad x \in \Omega, \quad B\phi = 0, \quad x \in \partial\Omega \tag{1.2}$$

Obviously, $U(x)$ is also a stationary solution of (1.1). However, as a solution of the delay equation (1.1), the stability of $U(x)$ is different.

The content of this paper is organized as follows. In Section 2 we show that when $b(x) > c(x)$, for any $\tau \geq 0$, the stationary solution $U(x)$ is globally asymptotically stable. In Section 3, for small τ and for any $b(x), c(x)$, it is given that $U(x)$ is linearized stable. Section 4 is devoted to the study of Hopf bifurcation from $U(x)$ as τ varies when $b(x) < c(x)$.

2. Globally Asymptotic Stability of $U(x)$ when $b(x) > c(x)$

Theorem 2.1 *Let* $L \equiv \max_{x\in\overline{\Omega}} \frac{c(x)}{b(x)}$ *and* $\tau > 0$. *If* $\lambda_1 < 1$ *and* $L < 1$, *then* $U(x)$ *is globally asymptotically stable in* (1.1) *with respect to every nonnegative initial function* $\eta(x,t)$ *with* $\eta(x,0) \equiv 0$.

Proof It is obvious that $c(x) \leq Lb(x)$ on $\overline{\Omega}$. Hence

$$c(x) \leq \frac{L}{L+1}(b(x) + c(x)), \ b(x) \geq \frac{1}{L+1}(b(x) + c(x)), \ \text{on} \ \overline{\Omega} \tag{2.1}$$

Let U^* be the nonnegative solution of the following parabolic problem

$$\frac{\partial U^*}{\partial t} - \frac{\partial^2 U^*}{\partial x^2} = U^*(a(x) - b(x)U^*), \quad \Omega \times [0,\infty)$$
$$BU^* = 0, \quad \partial\Omega \times [0,\infty)$$
$$U^*(x,0) = \eta(x,0), \quad \Omega$$