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EXISTENCE RESULTS FOR THE POSITIVE SOLUTIONS OF A FOURTH ORDER NONLINEAR EQUATIONS ON THE HEISENBERG GROUP*

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Abstract In this paper, we give some existence results for a fourth order nonlinear subelliptic equations on the Heisenberg group by the Leray-Schauder degree.
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1. Introduction

We are concerned with a fourth order nonlinear subelliptic problem

$$\begin{cases} \Delta^2_{H^n}u + c\Delta_{H^n}u = f((z,t),u) & \text{in } D \\ u|_{\partial D} = \Delta_{H^n}u|_{\partial D} = 0 \end{cases} \tag{1.1}$$

where D is a bounded open subset of the Heisenberg group H^n and Δ_{H^n} is the subelliptic Laplacian on H^n. The vector fields

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \end{aligned} \qquad j = 1, 2, \cdots, n \tag{1.2}$$

generate the real Lie algebra of left-invariant vector fields on H^n, see [1, 2].

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The subelliptic Laplacian is defined as

$$\Delta_{H^n} = \sum_{j=1}^{n}[X_j^2 + Y_j^2]$$

Δ_{H^n} is invariant w.r.t. left-translations.

In a recent paper, Garofalo and Lanconelli [2] introduced the problem

$$\begin{cases} \Delta_{H^n} u + f(u) = 0 & \text{in } D \\ u|_{\partial D} = 0 \end{cases} \tag{1.3}$$

They showed by the Mountain Pass theorem that if $f(u) = 0(|u|^{(Q+2)/(Q-2)})$ as $|u| \to \infty$, then (1.3) has a solution u. Zhang [3], by using Leray-Schauder degree, has proved that positive weak solutions to (1.3) do exist when $f((z,t),s) : D \times \boldsymbol{R} \to \boldsymbol{R}_+$ satisfying Caratheodory's continuity condition and growth restriction

$$0 \leq f((z,t),s) \leq a|s| \leq c_1 + c_2|s|^\alpha$$

It is of interest to note that Δ_{H^n} is not elliptic, we can't discuss the existence results for the problem (1.1) by the degree theory in H_0^1, where H_0^1 is the usual Sobolev spaces, but it can be solved in $S_1^{0,2}(D)$ by the degree theory, where $S_1^{0,2}(D)$ is Folland and Stein's Sobolev space. The nonlinear equation such as that in (1.1) may arise as Euler equations in some variational problems related to the geometry of CR manifolds, see [4–6]. In this paper we investigate the existence of the solution of a fourth order nonlinear equation (1.1). In Section 2, we give the results about the degree and review some relevant lemmas. The main results are considered and discussed in Section 3.

2. Preliminaries

Let X and Y be two Banach spaces and let $K \subseteq X$ be a cone in X and $\Omega \subset K$ be a bounded open subset in K. Suppose that $B : K \to Y$ and $F : \overline{\Omega} \to Y$ are two mappings. We assume that B and B^{-1} are continuous and $B(0) = 0$. We also assume that F is completely continuous and that $F(\overline{\Omega}) \subseteq B(K)$. Let $f = B - F$ and let $0 \in Y \backslash f(\partial\Omega)$. Then we have

Lemma 2.1 *Let* $\deg_B(f,\Omega,0) = \deg(I - B^{-1}F,\Omega,0)$, *where* $\deg(I - B^{-1}F,\Omega,0)$ *is the Leray-Schauder degree. Then the* $\deg_B(f,\Omega,0)$ *has the following properties:*

(B_1) *(Normality)* $\deg_B(B,\Omega,0) = 1$ *if* $0 \in \Omega$.

(B_2) *(Additivity)* $\deg_B(f,\Omega,0) = \deg_B(f,\Omega_1,0) + \deg_B(f,\Omega_2,0)$ *provided that* Ω_1, Ω_2 *are disjoint relatively open subsets of* D *such that* $0 \notin f(\Omega\backslash(\Omega_1 \cup \Omega_2))$.

(B_3) *(Solvability)* $Bx = Fx$ *has a solution in* Ω *if* $\deg_B(f,\Omega,0) \neq 0$.

(B_4) *(Excision)* $\deg_B(f,\Omega,0) = \deg_B(f,\Omega\backslash\omega,0)$ *if* $\omega \subseteq \overline{\Omega}$ *is closed and* $0 \notin f(\omega)$.