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ON THE RADIAL GROUND STATE OF P-LAPLACIAN EQUATION INVOLVING SUPER-CRITICAL OR CRITICAL EXPONENTS

Xuan Benjin* and Chen Zuchi
(Department of Mathematics, University of Science and Technology of China, Hefei 230026, China)
(E-mail: bjxuan@ustc.edu.cn; chenzc@ustc.edu.cn)
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Abstract In this paper, we consider the existence and uniqueness of the radial ground state to the following p-Laplacian equation involving super-critical or critical exponents: $\Delta_p u + u^q - |Du|^\sigma = 0$, $x \in R^n$, $2 \le p < n$, $q \ge [n(p-1)+p]/(n-p)$, $\sigma > 0$. Applying the shooting argument, the Schauder's fixed point theorem and some delicate estimates of auxiliary functions, we study the influence of the parameters n, p, q, σ on the existence and uniqueness of the radial ground state to the above p-Laplacian equation.
Key Words p-Laplacian equation; super-critical exponents; critical exponents; radial ground state; shooting argument.
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1. Introduction

In this paper, we consider the existence and uniqueness of the radial ground state to the following p-Laplacian equation involving super-critical or critical exponents:

$$\Delta_p u + u^q - |Du|^\sigma = 0, \quad x \in R^n \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $n > p \ge 2$, $q \ge [n(p-1)+p]/(n-p)$, $\sigma > 0$. By definition, a function $u(x)$ is said to be the radial ground state to (1.1), if $u(x) = u(r)$ satisfies:

1) $u(r), |u'|^{p-2}u' \in C^1([0,+\infty))$;

2) $u(r)$ is the solution of the following Cauchy problem:

$$\begin{cases} (|u'|^{p-2}u')' + \dfrac{n-1}{r}|u'|^{p-2}u' + u^q - |u'|^\sigma = 0, & r > 0 \\ u'(0) = 0, \quad u(0) = \xi > 0 \end{cases} \tag{1.2}$$

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3) $u(r)$ satisfies

$$u(r) > 0 \quad \text{for all } r > 0 \tag{1.3}$$

We are mainly interested in the influence of the parameters n, p, q, σ and ξ appearing in (1.2) on the existence and uniqueness of the radial ground state to (1.1). Many authors (cf. [1–6] and the references therein) have studied the following problem which is relevant to the problem (1.1)

$$\Delta_p u + f(x, u) = 0, \quad x \in R^n \tag{1.4}$$

At the same time, many authors (cf. [7] and the reference therein) are interested in the existence and uniqueness of the radial solution to (1.4) in a bounded domain, e.g. $B_R = \{x \in R^n; |x| \leq R, R > 0\}$.

In the view of the variational method, the exponent $q+1 = p^* = np/(n-p)$ in (1.1) is critical since Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ is not compact when $q+1 \geq p^* = np/(n-p)$. This noncompactness makes the functional corresponding to the equation (1.1) do not satisfy Palais-Smale condition, so when $q \geq p^* - 1$, the variational method is no longer valid. Papers [2] and [8], using the estimates of some auxiliary functions derived from the equation directly, deal with the existence, uniqueness and nonexistence of the radial ground state to (1.1) in the case where $p = 2$. In this paper, similar to [2], [7] and [8], based on some delicate *a-priori* estimates of auxiliary functions derived from the equation and the general Pohozaev-Pucci-Serrin integral identity, the existence and uniqueness of radial ground state to the problem (1.1), when $p \geq 2$, are studied. It is useful to study the problem (1.2) with the following boundary condition:

$$u(r) > 0 \quad \text{for} \quad 0 \leq r < R, \quad u(R) = 0 \tag{1.5}$$

Suppose that $u(r)$ is a solution to the problem (1.2) with boundary condition (1.3) or (1.5), we define the following auxiliary function on interval $(0, +\infty)$ or $(0, R)$:

$$H(r) = \frac{p-1}{p}|u'|^p + \frac{1}{q+1}u^{q+1} \tag{1.6}$$

$$G(r) = H(r) + \frac{\alpha u(r)|u'|^{p-2}u'}{r} \tag{1.7}$$

$$Z(r) = r^k G(r) - \beta r^k u|u'|^\sigma \tag{1.8}$$

$$S(r) = S_\gamma(r) = u^q - p|u'|^\gamma \tag{1.9}$$

where α, β, γ, k are all positive constants. Using some estimates of the above auxiliary functions, we obtain some results of the existence, uniqueness and nonexistence of the ground state to the problem (1.1).

This paper is organized as follows. In Section 2, we give some estimates of auxiliary functions H, G, Z, S, which play an important role throughout this paper. Section 3 deals with the existence and uniqueness of the local solution to the problem (1.2) for