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A NEW VISCOUS REGULARIZATION OF THE RIEMANN PROBLEM FOR BURGERS' EQUATION*

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Abstract This paper gives a new viscous regularization of the Riemann problem for Burgers' equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ with Riemann initial data $u = u_-(x \le 0), u = u_+(x > 0)$ at $t = 0$. The regularization is given by $u_t + \left(\frac{u^2}{2}\right)_x = \varepsilon e^t u_{xx}$ with appropriate initial data. The method is different from the classical method, through comparison of three viscous equations of it. Here it is also shown that the difference of the three regularizations approaches zero in appropriate integral norms depending on the data as $\varepsilon \to 0_+$ for any given $T > 0$.
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1. Introduction

The objective of this paper is to investigate a new regularization of scalar conservation laws

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \quad \text{where } u : R \times R^+ \to R \tag{1}$$

with Riemann initial data

$$u(x,0) = \begin{cases} u_-, & x \le 0 \\ u_+, & x > 0 \end{cases} \tag{2}$$

Our viscous equation is

$$\omega_t + \left(\frac{\omega^2}{2}\right)_x = \varepsilon e^t \omega_{xx} \tag{3}$$

This equation is derived from the unsteady Navier-Stokes equation

$$U_t + C(U) \cdot U_x = Re^{-1} D_1(U) + x_p^{-1} D_2(U, x) \tag{4}$$

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which descripts one dimensional unsteady gas motion with cylindrical symmetry, where

$$U=\begin{pmatrix}\rho\\ v\\ \sigma\end{pmatrix}, C(U)=\begin{pmatrix}\varepsilon v & 1+\varepsilon\rho & 0\\ \dfrac{1+\varepsilon T}{1+\varepsilon\rho} & \varepsilon v & 1+\varepsilon T\\ 0 & 0 & \varepsilon v\end{pmatrix}$$

$$D_1(U)=\begin{pmatrix}0 & 0 & 0\\ 0 & \dfrac{4}{3}+\nu & 0\\ \dfrac{\gamma-1}{P_r} & 0 & \dfrac{\gamma}{P_r}\end{pmatrix}U_{xx}, D_2(U)=-\frac{jx_p}{x}\begin{pmatrix}[1+\varepsilon\rho]\\ 0\\ 0\end{pmatrix}$$

See [Eq.(3.86),1] or [Chapter iv Eq.(21),2]. (3) is a perturbation of Eq. (4) in the form $U=U_c+\varepsilon(U_0+\varepsilon U_1+\cdots+\mu_1U_2+\mu_2U_3+\cdots)$, where $\mu_1=Re^{-1},\mu_2=x_p^{-1}$. The situation is retrieved by introducing dependence of the solution on the slow variables

$$\tau_1=\varepsilon t,\quad \tau_2=\mu_1 t,\quad \tau_3=\mu_2 t$$

besides the fast variable $t=\tau_0$. Thus, we have, formally

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial\tau_0}+\varepsilon\frac{\partial}{\partial\tau_1}+\mu_1\frac{\partial}{\partial\tau_2}+\mu_2\frac{\partial}{\partial\tau_3}$$

In this situation it is obtained that a singular equation

$$W_t+WW_x+\frac{W}{2t}=\varepsilon W_{xx} \tag{5}$$

where x,t are new space-time variable and W is a scalar. Then using translation $\omega=tW,t'=\ln t$ and replacing t' with t we can get (3). The detial can be found in [2]. Equation (3) is also the model of the propagation of finite-amplitude sound waves in variable-area (See [3,4]), where u is an acoustic variable, with the linear effects of changes in the duct area taken out, and εe^t depends on the particular duct chosen.

The classical regularization of (1) (2) is to imbed (1) in the viscous equation

$$u_t+\left(\frac{u^2}{2}\right)_x=\varepsilon u_{xx} \tag{6}$$

The difficulty with this regularization is that (6) does not possess space-time dilational invariance $((x,t)\to(\alpha x,\alpha t),\alpha>0)$ of (1) (2). In order to overcome this difficulty Dafermos [5], Kalasnikov [6], Tupciev [7] independently suggested the viscous regularization of (1) (2) given by

$$v_t+\left(\frac{v^2}{2}\right)_x=\varepsilon t v_{xx} \tag{7}$$

$$v(x,0)=\begin{cases}u_-, & x\le 0\\ u_+, & x>0\end{cases} \tag{8}$$